Journal of
Data Acquisition and Processing
ISSN: 1004-9037
https://sjcjycl.cn/
DOI: 10.5281/zenodo.776475

# APPLICATION OF HOMOTOPY ANALYSIS METHOD FOR SOLVING NONLINEAR PROBLEMS

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**ABSTRACT**

Many models used to explain physical events are written in the form of a nonlinear differential equation. Semi-analytical methods have been developed, such as the homotropy analysis method (HAM), to solve nonlinear ordinary/partial differential equations; on the other hand, non-perturbative methods have been developed to investigate physical problems that do not require a small physical parameter to be used as the perturbation parameter. The auxiliary parameter, which is stored in the HAM, allows for straightforward regulation of the series solutions' convergent area. This analytic strategy is used to get the precise answers to linear-algebra problems. The results demonstrate the efficacy and ease of the suggested approach.

**Keywords:** Analysis, Method, Homotopy, Nonlinear, Parameters, Linear, Algebra

## 1. INTRODUCTION

Many models used to explain physical events are written in the form of a nonlinear differential equation. Unfortunately, only a small fraction of these equations has simple, straightforward solutions. As a result, approximation methods have been developed to carry out approximations to these differential equations. The Perturbation methods are an example of such a method. The presence of small/large parameters, the so-called perturbation quantity, is the foundation of perturbation methods. Unfortunately, many nonlinear scientific and engineering issues do not include perturbation quantities of this kind. The artificial small parameter approach, the -expansion method, and the Adomian's decomposition method are all examples of non-perturbative procedures. These nonperturbative approaches do not rely on large values of parameters, as is the case with perturbation techniques. The convergence area and rate of a particular approximation series cannot be easily adjusted or controlled using either perturbation techniques or nonperturbative methods alone.

It is well known that nonlinear partial differential equations can be used to describe a wide range of phenomena in a number of different scientific disciplines, including but not limited to physics (where they have been applied to study things like magneto fluid dynamics, water surface gravity waves, electromagnetic radiation reactions, and ion acoustic waves in plasma). It's not only physics; very few issues in the natural sciences have straightforward solutions. Consequently, it is common practise to first investigate an ideal model, which is selected to reflect as much of the natural of the real scientific system as possible as an appropriate approximation, and then to deal with other effects using some efficient perturbative and/or non perturbative techniques. When studying physical systems that can be solved accurately but have tiny perturbation parameters, perturbation theory (PT) is often used; this involves expansion around the perturbation parameter, with approximants given as power series of the parameters under study. However, non-perturbative approaches have been developed for investigating physical issues whereby a tiny physical parameter cannot be employed as the perturbation parameter. Applying optimal perturbation theory, which does not rely on parameters,

As a semi-analytical approach, the homotopy analysis method (HAM) may be used to solve nonlinear ordinary/partial differential equations. Using the homotopy idea from topology, the homotopy analysis approach produces a convergent series solution for nonlinear systems. To accommodate the system's nonlinearities, a homotopy-Maclaurin series is used to achieve this goal. In his doctoral dissertationfrom 1992, Liao Shijun of Shanghai Jiaotong University introduced the HAM, which was later modified in 1997 to incorporate a non-zero auxiliary parameter known as the convergence-control parameter, c0, to create a homotopy on a differential system in general form. [Convergence may be easily checked and enforced by using a non-physical variable called the convergence-control parameter. In contrast to most analytical and semi-analytic methods for solving nonlinear partial differential equations, the HAM may demonstrate convergence of the series solution intuitively.

## 2. LITERATURE REVIEW

**Rasha Fahad et.al (2019)** This study applies the traditional homotopy analysis technique to second-order initial value problems including various discontinuities, both linear and nonlinear. Numerical findings were compared with the precise solution using the usual homotopy analysis approach, which included iterating the integral equation and the numerical solution using the Simpson rule. The projected sequence of convergence and the maximum absolute error, relative error, residual error, and residual error were also provided. I think the study has merit and think it should be published in a peer-reviewed publication.

**Inyama Simeon Chioma et.al (2019)** In this study, we provide a novel semi-analytic approach, the Homotopy Analysis Method (HAM), to solving the SEIRS Epidemic Mathematical Model via a series of model modifications. We initially created and properly assessed the modified SEIRS model. By creating the invariant area and demonstrating the positivity of the solutions, we were able to get insight into the model's fundamental characteristics. After calculating the fundamental reproduction number, we were able to get the stable states of disease-free equilibrium (DFE) and endemic equilibrium (EE), respectively. Additionally, we used the Lyaponuv approach to demonstrate the worldwide robustness of the endemic equilibrium, and we used the HAM to quickly and precisely solve the model. And last, using computing tools like MAPLE 15, a numerical solution (simulation) of the model was achieved.

**Jamal Oudetallah et.al (2021)** In order to provide numerical solutions for nonlinear initial value problems, this article proposes a variant of a well-known homotopy analysis method called Quotient Homotopy Analysis Method, or QHAM for short. This method's foundation rests on the construction of a suitable homotopy equation, the solution to which is established as the quotient of two formulated power series; this, in turn, forms the basis stone of the general numerical solution for the nonlinear problems. The significance and potential role of the established method are revealed through two numerical comparisons between the exact solution and the approximate numerical solution using two examples.

**Atanaska Georgieva (2022)** An approximation approach for solving a nonlinear Volterra-Fredholm fuzzy integro-differential equation is presented (NVFFIDE). The homotopy analysis technique is used (HAM). An exact example of the examined problem is transformed into a nonlinear system of Volterra-Fredholm integro-differential equations. An estimate for the fuzzy solution of the NVFFIDE is shown, which is produced with the assistance of HAM. It is shown that the suggested approach converges. Validity and utility of the suggested method are shown using a numerical example.

**Pankaj Dumkaet.al (2022)** Here, the Homotopy and Perturbation approach is used to the simultaneous solution of linear and nonlinear, steady and unsteady heat transport equations (HPM). As an added bonus, HPM has been implemented using the help of the Python module SymPy, which is used to do symbolic computations. There was a total of three issues addressed here: heat transport in a uniform rectangular fin by radiation, steady-state conduction with heat production, and lumped capacitance analysis with a changeable specific heat of the material. The HPM has consistently outperformed its analytical and numerical counterparts. SymPy's execution has been outlined, and a step-by-step guide on using Python to create HPM has been provided for all three use cases.

## 3. METHODOLOGY & DATA ANALYSIS

### Homotopy Analysis Method

The following differential equation serves to illustrate the HAM concept.

$$N\left[u(x,t)\right]=0,$$

the unknown function u is represented by the notation N (x, t), where x and t are the independent variables. To keep things simple, we don't account for any boundary or starting circumstances that are equivalently handled. At first, we use the HAM to formulate the so-called zeroth-order deformation equation.

$$(1-q)\,L\left[\phi(x,t;q)-u_0(x,t)\right]=q\;\hbar\;H(x,t)\;N\left[\phi(x,t;q)\right],$$

The embedding parameter is denoted by q, the auxiliary parameter by 6= 0, the auxiliary linear operator by L, the unknown function by (x, t; q), the initial guess by u0(x, t), and the nonzero auxiliary function by H (x, t). With q = 0 and q = 1, it is easy to see that equation simplifies to

$$\phi(x,t;0) = u_0(x,t), \quad \phi(x,t;1) = u(x,t),$$

respectively. Therefore, the solution shifts from the initial estimate u0(x, t) to the answer u(x, t) when q goes from 0 to 1. If one takes the Taylor series expansion of f(x, t), q, one obtains

$$\phi(x,t;q) = u_0(x,t) + \sum_{\mathrm{m}=1}^{\infty} u_m(x,t) q^m,$$

Where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;q)}{\partial q^m}|q = 0.$$

The value of the optional parameter h determines whether or not the series converges. Given that it converges at q = 1, one obtains

$$u(x,t) = u_0(x,t) + \sum_{\mathrm{m}=1}^{\infty} u_m(x,t),$$

which, as Liao demonstrated, must be one of the answers to the original nonlinear equation. Create some vector definitions

$$\vec{u_n} = \left(u_0(x,t), u_1(x,t), \ldots, u_n(x,t)\right).$$

Obtain the mth derivative of the zeroth-order deformation equation with respect to q, and then divide by m. The m th order deformation equation is obtained by setting q to zero.

$$L\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] = \hbar\ \Re_m(\vec{u_{m-1}}),$$

Where

$$\Re_m(\vec{u_{m-1}}) = \frac{1}{m!}\frac{\partial^{m-1} N[\phi(x,t;q)]}{\partial q^{m-1}}|q = 0,$$

And

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1. \end{cases}$$

It is important to note that for m 1, um(x, t) is governed by a linear equation with linear boundary conditions derived from the original problem, which can be solved using symbolic computation software like Mathematica or Maple. For more information on how the above approach converges, see Liao. If the equation can only have one solution, then this process will give you that one. In the case that the equation does not have a unique solution, the HAM will return a solution from a set of candidates.

**Applications**

The following illustrations will be used to show how HAM can be used:

**Example 1**

Think about the following equation for a moment.

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) u = -\left(\frac{1}{3}\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{1}{6}\frac{\partial^2 u}{\partial t^2}\right)^3 - 16.u,$$

to do with the way things started off

$$u(x,0) = -x^4,\ \frac{\partial u}{\partial t}(x,0) = 0,$$

And

$$\frac{\partial^2 u}{\partial t^2}(x,0) = 0.$$

Now, let's consider the following linear operator as an example of one that can be solved using homotopy analysis:

$$L\left[\phi(x,t;q)\right] = \frac{\partial^3 \phi(x,t;q)}{\partial t^3},$$

featuring the quality that

$$L\left[\phi(x,t;q)\right] = \frac{\partial^3 \phi(x,t;q)}{\partial t^3},$$

means that

$$L^{-1}(.) = \int_0^t \int_0^t \int_0^t (.)\ dt\ dt\ dt,$$

The nonlinear operator is now defined as

$$\begin{aligned} N\left[\phi(x,t;q)\right] &= \frac{\partial^3\phi(x,t;q)}{\partial t^3} - \frac{\partial^3\phi(x,t;q)}{\partial t\partial x^2} - \frac{\partial^4\phi(x,t;q)}{\partial t^2\partial x^2} + \frac{\partial^4\phi(x,t;q)}{\partial x^4} \\ &+ \frac{1}{216}\left(\frac{\partial^2\phi(x,t;q)}{\partial t^2}\right)^3 - \frac{1}{9}\left(\frac{\partial^2\phi(x,t;q)}{\partial x^2}\right)^2 + 16\phi(x,t;q). \end{aligned}$$

The zero-order deformation equation is derived from the above definition by

$$(1-q)\,L\left[\phi(x,t;q) - u_0(x,t)\right] = q\,\hbar\,H(x,t)\ N\left[\phi(x,t;q)\right],$$

where q ∈ [0, 1] is the embedding parameter, 6= 0 is an optional parameter, L is a linear operator that helps out when things get complicated, (x, t; q) is a mystery function, u0(x, t) is a first guess, and H(x, t) is an auxiliary function that doesn't start out at zero. When the embedding parameter q is set to 0 and 1 it is easy to see that the equation becomes

$$\phi(x,t;0) = u_0(x,t), \quad \phi(x,t;1) = u(x,t),$$

The equation for the deformation of m th-order is then obtained.

$$\begin{aligned} L\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] &= \hbar\Re_m(\vec{u_{m-1}}) \\ &\Longrightarrow \\ u_m(x,t) &= \chi_m u_{m-1}(x,t) + \hbar L^{-1}\left(H(x,t)\Re_m(\vec{u_{m-1}})\right), \end{aligned}$$

Where

$$\begin{aligned} \Re_m(\vec{u_{m-1}}) &= \frac{\partial^3 u_{m-1}}{\partial t^3} - \frac{\partial^3 u_{m-1}}{\partial t\partial x^2} - \frac{\partial^4 u_{m-1}}{\partial t^2\partial x^2} + \frac{\partial^4 u_{m-1}}{\partial x^4} + \frac{1}{9}\sum_{i=0}^{m-1}\frac{\partial^2 u_i}{\partial x^2} \\ &- \frac{1}{216}\sum_{i=0}^{m-1}\sum_{j=0}^{m-1-i}\frac{\partial^2 u_i}{\partial t^2}\frac{\partial^2 u_j}{\partial t^2}\frac{\partial^2 u_{m-j-i-1}}{\partial t^2} + 16u_{m-1}, \end{aligned}$$

Find the answer to the following problem using the given starting data.

$$u_m(x,0) = 0\ and\ \frac{\partial u_m}{\partial t}(x,0) = 0.$$

For the sake of clarity, let's take u0(x, t) = −x 4 and

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar \int_0^t \int_0^t \int_0^t \left( \frac{\partial^3 u_{m-1}}{\partial t^3} - \frac{\partial^3 u_{m-1}}{\partial \xi_1 \partial x^2} - \frac{\partial^4 u_{m-1}}{\partial \xi_1^2 \partial x^2} + \frac{\partial^4 u_{m-1}}{\partial x^4} \right) d\xi_1 \; d\xi_2 \; dt$$

$$+\hbar \int_0^t \int_0^t \int_0^t \frac{1}{9} \sum_{i=0}^{m-1} \frac{\partial^2 u_i}{\partial x^2} \frac{\partial^2 u_{m-1-i}}{\partial x^2} d\xi_1 \; d\xi_2 \; dt$$

$$-\hbar \int_0^t \int_0^t \int_0^t \left( \frac{1}{216} \sum_{i=0}^{m-1} \sum_{j=0}^{m-1-i} \frac{\partial^2 u_i}{\partial t^2} \frac{\partial^2 u_j}{\partial t^2} \frac{\partial^2 u_{m-j-i-1}}{\partial t^2} - 16 u_{m-1} \right) d\xi_1 \; d\xi_2 \; dt,$$

The remaining parts are provided by

$$\begin{aligned} u_1(x,t) &= 0 \\ u_2(x,t) &= 0 \\ u_3(x,t) &= -\hbar \; 4 \; t^3 \\ u_4(x,t) &= u_5(x,t) = ... = 0. \end{aligned}$$

Thus, an approximate solution can be found at h= −1

$$u(x,t) = -x^4 + 4 \; t^3$$

which is also an exact solution and was obtained by A.Roozi et al.

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) u = \left(\frac{\partial^2 u}{\partial t^2}\right)^2 - \left(\frac{\partial^2 u}{\partial x^2}\right)^2 - 2 \; u^2,$$

to do with the way things started off

$$u(x,0) = e^x, \; \frac{\partial u}{\partial t}(x,0) = e^x \text{ and } \; \frac{\partial^2 u}{\partial t^2}(x,0) = e^x,$$

The nonlinear operator is now defined as

$$\begin{aligned} N\left[\phi(x,t;q)\right] &= \frac{\partial^3 \phi(x,t;q)}{\partial t^3} - \frac{\partial^3 \phi(x,t;q)}{\partial t \partial x^2} - \frac{\partial^4 \phi(x,t;q)}{\partial t^2 \partial x^2} + \frac{\partial^4 \phi(x,t;q)}{\partial x^4} \\ &\quad + \left(\frac{\partial^2 \phi(x,t;q)}{\partial t^2}\right)^2 - \left(\frac{\partial^2 \phi(x,t;q)}{\partial x^2}\right)^2 + 2\phi^2(x,t;q), \end{aligned}$$

The equation for the deformation of mth order is thus obtained.

$$L\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] = \hbar \; \Re_m(\vec{u_{m-1}}),$$

Where

$$\Re_m(\vec{u}_{m-1}) = \frac{\partial^3 u_{m-1}}{\partial t^3} - \frac{\partial^3 u_{m-1}}{\partial t \partial x^2} - \frac{\partial^4 u_{m-1}}{\partial t^2 \partial x^2} + \frac{\partial^4 u_{m-1}}{\partial x^4} + \sum_{i=0}^{m-1} \frac{\partial^2 u_i}{\partial x^2}\frac{\partial^2 u_{m-1-i}}{\partial x^2} - \sum_{i=0}^{m-1} \frac{\partial^2 u_i}{\partial t^2}\frac{\partial^2 u_{m-1-i}}{\partial t^2} + 2\sum_{i=0}^{m-1} u_i u_{m-1-i},$$

then the equation for the linear differential is

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar \int_0^t \int_0^t \int_0^t \left( \frac{\partial^3 u_{m-1}}{\partial t^3} - \frac{\partial^3 u_{m-1}}{\partial \xi_1 \partial x^2} - \frac{\partial^4 u_{m-1}}{\partial \xi_1^2 \partial x^2} + \frac{\partial^4 u_{m-1}}{\partial x^4} \right) d\xi_1 \, d\xi_2 \, dt$$
$$+\hbar \int_0^t \int_0^t \int_0^t \left( \sum_{i=0}^{m-1} \frac{\partial^2 u_i}{\partial x^2}\frac{\partial^2 u_{m-1-i}}{\partial x^2} - \sum_{i=0}^{m-1} \frac{\partial^2 u_i}{\partial t^2}\frac{\partial^2 u_{m-1-i}}{\partial t^2} + 2\sum_{i=0}^{m-1} u_i \, u_{m-1-i} \right) d\xi_1 \, d\xi_2 \, dt,$$

First, we use the rough estimate.

$$u_0(x,t) = \left(1 + t + \frac{t^2}{2}\right) e^x,$$

Find the answer to the above equation using the given starting data.

$$u_m(x,0) = 0 \ and \ \frac{\partial u_m}{\partial t}(x,0) = 0,$$

Find the answer to the above equation using the given starting data.

$$u_m(x,0) = 0 \ and \ \frac{\partial u_m}{\partial t}(x,0) = 0,$$

we get

$$u_1(x,t) = -\frac{\hbar \, t^3 \, e^x}{6}$$
$$u_2(x,t) = -\frac{\hbar^2 \, t^4 \, e^x}{24}$$
$$u_3(x,t) = -\frac{\hbar^3 \, t^5 \, e^x}{120},$$

hence, when h = 1, we get an approximation to the solution:

$$u(x,t) = e^{x+t},$$

It is also the same as the precise answer found by A.Roozi et al.

**Example 3**

Think about the following equation for a moment.

$$\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right)u=u\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial x},$$

to do with the way things started out

$$u(x,0)=cos(x),\ \frac{\partial u}{\partial t}(x,0)=-sin(x)\ and\ \frac{\partial^2 u}{\partial t^2}(x,0)=-cos(x),$$

to start, let's define the nonlinear operator:

$$\begin{aligned}N\left[\phi(x,t;q)\right] \;=\; & \frac{\partial^3\phi(x,t;q)}{\partial t^3}-\frac{\partial^3\phi(x,t;q)}{\partial t\partial x^2}-\frac{\partial^4\phi(x,t;q)}{\partial t^2\partial x^2}+\frac{\partial^4\phi(x,t;q)}{\partial x^4}\\ & -\phi(x,t;q)\frac{\partial\phi(x,t;q)}{\partial t}-\frac{\partial^2\phi(x,t;q)}{\partial t^2}\frac{\partial\phi(x,t;q)}{\partial x}.\end{aligned}$$

And from this, we can derive the m th-order deformation equation:

$$L\left[u_m(x,t)-\chi_m u_{m-1}(x,t)\right]=\hbar\ \Re_m(\vec{u_{m-1}}),$$

Where

$$\begin{aligned}\Re_m(\vec{u_{m-1}}) \;=\; & \frac{\partial^3 u_{m-1}}{\partial t^3}-\frac{\partial^3 u_{m-1}}{\partial t\partial x^2}-\frac{\partial^4 u_{m-1}}{\partial t^2\partial x^2}+\frac{\partial^4 u_{m-1}}{\partial x^4}+\sum_{i=0}^{m-1}u_i\frac{\partial u_{m-1-i}}{\partial t}\\ & -\sum_{i=0}^{m-1}\frac{\partial^2 u_{m-1}}{\partial t^2}\frac{\partial u_{m-1-i}}{\partial x},\end{aligned}$$

It is necessary to start with a preliminary estimate.

$$u_0(x,t)=cos(x)-tsin(x)-\left(\frac{t^2}{2}\right)sin(x),$$

And

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar \int_0^t \int_0^t \int_0^t \left( \frac{\partial^3 u_{m-1}}{\partial t^3} - \frac{\partial^3 u_{m-1}}{\partial \xi_1 \partial x^2} - \frac{\partial^4 u_{m-1}}{\partial \xi_1^2 \partial x^2} + \frac{\partial^4 u_{m-1}}{\partial x^4} \right) d\xi_1 \, d\xi_2 \, dt$$
$$+\hbar \int_0^t \int_0^t \int_0^t \left( -u_{m-1} \frac{\partial u_{m-1}}{\partial t} - \frac{\partial^2 u_{m-1}}{\partial t^2} \frac{\partial u_{m-1}}{\partial x} \right) d\xi_1 \, d\xi_2 \, dt,$$

Find the answer to the above equation based on the information provided.

$$u_m(x,0) = 0 \ and \ \frac{\partial u_m}{\partial t}(x,0) = 0,$$

we get

$$\begin{aligned} u_1(x,t) &= -\frac{\hbar \, t^3 \, sin(x)}{6} - \frac{\hbar \, t^4 \, cos(x)}{24} \\ u_2(x,t) &= -\frac{\hbar^2 \, t^5 \, sin(x)}{120} - \frac{\hbar^2 \, t^6 \, cos(x)}{720} \\ u_3(x,t) &= -\frac{\hbar^3 \, t^7 \, sin(x)}{5040} - \frac{\hbar^2 \, t^8 \, cos(x)}{40320} \end{aligned}$$

therefore, we have a solution approximation at h = 1:

$$u(x,t) = cos(x+t),$$

which is also an exact solution and was obtained by A.Roozi et al.

**Example 4**

Think about the following equation for a moment.

$$\left( \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) u = \frac{\partial u}{\partial t} - 2u,$$

to do with the way things started off

$$u(x_1,x_2,0) = sinh(x_1+x_2), \ \frac{\partial u}{\partial t}(x_1,x_2,0) = 2sinh(x_1+x_2),$$

And

$$\frac{\partial^2 u}{\partial t^2}(x_1,x_2,0) = 4sinh(x_1+x_2),$$

The nonlinear operator is now defined as

$$N\left[\phi(x,t;q)\right] = \frac{\partial^3\phi(x_1,x_2,t;q)}{\partial t^3} - \frac{\partial^3\phi(x_1,x_2,t;q)}{\partial t\partial x_1^2} - \frac{\partial^3\phi(x_1,x_2,t;q)}{\partial t\partial x_2^2} - \frac{\partial^4\phi(x_1,x_2,t;q)}{\partial x_1^2\partial t^2}$$
$$+\frac{\partial^4\phi(x_1,x_2,t;q)}{\partial x_1^4} + \frac{\partial^4\phi(x_1,x_2,t;q)}{\partial x_1^2\partial x_2^2} - \frac{\partial^4\phi(x_1,x_2,t;q)}{\partial t^2\partial x_2^2} + \frac{\partial^4\phi(x_1,x_2,t;q)}{\partial x_1^2\partial x_2^2} + \frac{\partial^4\phi(x_1,x_2,t;q)}{\partial x_2^4}$$
$$-\frac{\partial\phi(x_1,x_2,t;q)}{\partial t} + 2\phi(x_1,x_2,t;q).$$

Using a rough estimate to begin

$$u_0(x_1,x_2,t) = \left(1+2t+2t^2\right) sinh(x_1+x_2)$$

and additional parts are

$$u_1(x_1,x_2,t) = -\frac{4\hbar\ t^3\ sinh(x_1+x_2)}{3}$$
$$u_2(x_1,x_2,t) = \frac{2\hbar^2\ t^4\ sinh(x_1+x_2)}{3}$$
$$u_3(x_1,x_2,t) = -\frac{4\hbar^3\ t^5\ sinh(x_1+x_2)}{15}$$

Therefore, the approximate solution is given by at h = 1.

$$u(x_1,x_2,t) = sinh(x_1+x_2)\ e^{2t}$$

and agrees with the exact solution found by A.Roozi et al.

## 4. CONCLUSION

The artificial small parameter approach, the -expansion method, and the Adomian's decomposition method are all examples of non-perturbative procedures. However, non-perturbative approaches have been developed for investigating physical issues whereby a tiny physical parameter cannot be employed as the perturbation parameter. Using the homotopy idea from topology, the homotopy analysis approach produces a convergent series solution for nonlinear systems. We may easily modify and regulate the convergence zone of solution series by selecting appropriate values for the auxiliary parameter, the auxiliary function H(t), and the auxiliary linear operator L, which is not possible with any other analytical approach. The first thing we can do is choose the initial estimates, the auxiliary parameter, the auxiliary function H(t), the auxiliary linear operator L, and so on. Second, it was shown that the HAM is a strong analytic-numeric system for addressing a wide range of nonlinear problems with little effort. Maple 13, a software programme, has performed the numerical calculation.

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