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GENERALIZED FOURIER TRANSFORM IN RIEMANN-LIOUVILLE SENSE AS A DENOISER FOR SIGNAL

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ABSTRACT

Signal processing is a vast applied area of research, and It is applicable in engineering, medicine, war, and many more fields. Many signal processing methods are already available like Fourier transform, fractional Fourier transforms, and many more. In this work, we defined a generalized Fourier transform and proposed a new technique for signal denoising: the Fourier transform of signal in the presence of fractional calculus. This proposed method is more convenient than previous methods because of its applicability in the fractional variable.By fractional variable, we reached the more appropriate signal.

Keywords: R-L Integrodifferential, Low pass filter, De-noising signal, Digital Signal Processing.

1. INTRODUCTION

Signal processing started as a numerical technique in the early 17th century, but it was a known subject in the 19th century. In 1948 Claud Shannon wrote a paper on communicationsmathematical theory [1]. His study proved to be a milestone in signal processing, and researchers started their work in this field. In 1996 Hall et al. wrote a book on SAGE- a Radar technology of the U.S.- an example of digital signal processing. The Fourier transformwas a widely used practical method in signal processing in the 20th century; In 2006, Salih published a book on the use of the Fourier transform in signal processing [2], after some time,the fractional Fourier transform, a generalization of the Fourier transform, was developed, Namias initially proposed the fractional Fourier change in 1980 [3], and the FRFT (Fractional Fourier Transform) defined by Namias is applicable in many problems of signal processing [4–6] and this definition move the researchers interest in the fractional domain. Presently many researchers are working in the field of signal processing using fractional calculus, viz., In 2000 Tseng wrote a paper on fractional calculus using Fourier transform [7], pulling the attention to the field of applicationof fractional calculus in signal processing with continuation to his work. In 2004 Margin and Richard found the application of fractional calculus in bioengineering [8]. Then in 2007,Assaleh et al. found the application of fractional calculus in their work on modeling speech signals [9]. In 2018 Cruz-Duarte et al. wrote a paper based on applying the Caputo-Fabrizio derivative. Recently, many more research papers have been in the queue of application of various definitions of FC in signal processing [10, 11]. In this research article, we will explorethe application of Reimann Liouville Fractional and Integration, called Integrodifferentiaon with Fourier transforms, to design a low-pass filter. In my work, I am using the MATLAB code FracUnif to find out the fractional derivative, which from the study of the following research articles [12–15]. My research work is inspired by the different mathematician's workas referenced in my paper [7, 16].

In this paper, first, we define the fractional derivative and integral of Riemann Liouville and the fractional derivative of Caputo; after this, we will find out the Fourier transform of the fractional Riemann-Liouville product of a function then we will find the fractional Riemann-Liouville integral of the inverse Fourier transform of a process. In the end, we will find out the Graph of the corrupted and denoised sinusoidal signal using various values of fractional parameter a between 0 to 1 by numerical simulation.

2. Preliminaries

In this section, we'll review a few terminologies and essential fractional calculus findings pertinent to the rest of the inquiry.

Definition 2.1. [17] Following is the definition of the fractional integral of a function ξ with order, $a > 0$ lower bound zero in Riemann-Lioville sense:

$$I^a \xi(t) = \frac{1}{\Gamma(a)} \int_0^t (t-\eta)^{a-1} \xi(\eta) d\eta, \ t > 0$$

and $I^0 \phi(t) := \phi(t)$, where the Euler Gamma function is $\Gamma(\cdot)$. For $b > 0$, this fractional integral satisfies the conditions $I^a \circ I^b = I^{a+b}$.

Definition 2.2. [18] Given is the function's Riemann-Liouville fractional derivative at the lower limit zero of order $a > 0$.

$$D^a \xi(t) = \frac{1}{\Gamma(n-a)} \frac{d^n}{dt^n} \int_0^t (t-\eta)^{n-a-1} \xi(\eta) d\eta$$

where $n-1 < a < n, n \in \mathbb{N}$, up to order $(n-1)$, an absolutely continuous derivative exists for the function $\xi(t)$. Moreover $D^0 \xi(t) = \xi(t)$ and $D^a I^a \xi(t) = \xi(t)$ for $t > 0$.

Definition 2.3. [18] The Caputo fractional derivative of a function $\xi \in \mathcal{C}^n([0,\infty))$ with the lower limit zero of order $a > 0$ is given by

$$^c D^a \xi(t) = \frac{1}{\Gamma(n-a)} \int_0^t (t-\eta)^{n-a-1} \frac{d^n}{d\eta^n} \xi(\eta) d\eta,$$

where $n-1 < a < n, n \in \mathbb{N}$.

Definition 2.4. [19] Let ξ be a function of class C, then its Fourier transform FT is defined as-

$$FT(\xi(t)) = \bar{\xi}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iwt} \xi(t) dt$$

Definition 2.5. [19] Let ξ be a function of class C, then its Inverse Fourier transform IFT is defined as-

$$\text{IFT}\ (\bar{\xi}(w)) = \xi(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iwt}\bar{\xi}(w)dt$$

3. Main Result

3.1. Fourier transform of derivative of $\xi(t)$

$$FT\left(\frac{d^n(\xi)}{dt^n}\right) = (-iw)^n\bar{\xi}(w) \tag{3. 1}$$

. Where function $\xi(t)$ and its nth order derivatives vanish at $\pm\infty$

3.2. Fourier transform of fractional derivative defined in (2.2) of $\xi(t)$

$$\begin{aligned} FT\big(D^a(\xi(t)) = \bar{\xi}_a(w) \quad &= FT\left(\frac{1}{\Gamma(n-a)}\frac{d^n}{dt^n}\int_0^t (t-\eta)^{n-a-1}\xi(\eta)d\eta\right) \\ &= \frac{1}{\Gamma(n-a)}FT\left(\frac{d^n}{dt^n}\int_0^t (t-\eta)^{n-a-1}\xi(\eta)d\eta\right)\Bigg) \end{aligned}$$

from (3.1) we find

$$\begin{aligned} FT\Bigg(D^a(\xi(t)) &= \frac{1}{\Gamma(n-a)}(-iw)^n FT\left(\left(\int_0^t (t-\eta)^{n-a-1}\xi(\eta)d\eta\right)\right), \\ &= \frac{1}{\Gamma(n-a)}(-iw)^n FT\big(t^{(n-a-1)} * \xi(t)\big), \\ &= \frac{1}{\Gamma(n-a)}(-iw)^n FT\big(t^{(n-a-1)}FT(\xi(t)\big), \\ &= \frac{1}{\Gamma(n-a)}(-iw)^n \Gamma(n-a)(-iw)^{(a-n)}\xi(w), \end{aligned}$$

Where $FT\big(t^{(n-a-1)} = \Gamma(n-a)(-iw)^{(a-n)}$, Hence

$$FT\big(D^a(\xi(t)) = \bar{\xi}_a(w) = (-iw)^a\bar{\xi}(w) \tag{3. 2}$$

3.3 Fractional order integration of inverse Fourier transforms for (3.2)

$$D^{-a}\text{IFT}\ \big(\bar{\xi}_a(w)\big) = D^{-a}\text{IFT}\ \big((-iw)^a\bar{\xi}(w)\big)$$

From (3.1) one can see that $(-iw)^a\bar{\xi}(w) = FT\left(\frac{d^n(\xi)}{dt^n}\right)$ then

$$D^{-a}\text{IFT}\left(\bar{\xi}_a(w)\right) = D^{-a}IFT\left(FT(D^a(\xi))\right),$$
$$= D^{-a}D^a(\xi(t)),$$
$$\Rightarrow D^{-a}\text{IFT}\left(\bar{\xi}_a(w)\right) = \xi(t). \tag{3.3}$$

3.4 Generalized Fourier transform

One can see from (3.2) the Fourier transform of fractional order derivative of function $\xi(t)$ can be defined as a generalized Fourier transforms as for fractional angle $a = 0$, it will become Fourier transform. Generalized Fourier transform will be notated as $\bar{\xi}_a(w)$ and defined as-

$$\bar{\xi}_a(w) = (-iw)^a \bar{\xi}(w) \tag{3.4}$$

3.5 Convolution of two functions in Generalized Fourier transform Environment

The convolution of two functions, f, and g, can be defined as-

$$f * g = (-iw)^a \int_{-\infty}^{\infty} f(t-u)g(u)du \tag{3.5}$$

3.6 Convolution theorem for Generalized Fourier transform

Theorem 3.1. If $h(t)$ is the convolution of two functions $f(t)$ and $g(t)$, then the generalized Fourier transform of $h(t)$ is equal to the multiplication of the generalized Fourier transform of $f(t)$ and $g(t)$ i. e.

$$\overline{h_a} = \bar{f}_a \overline{g_a}$$

, Where $\overline{h_a}$ is fractional Fourier transform of $h(t)$.

Proof. Given that-

$$h(t) = f(t) * g(t) = (-iw)^a \int_{-\infty}^{\infty} f(t-u)g(u)du$$

Then

$$\overline{h_a} = (-iw)^a \int_{-\infty}^{\infty} e^{iwt} h(t)dt,$$
$$= (-iw)^a(-iw)^a \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{iwt} f(t-u)g(u)dudt$$

Choose $t - u = x$, Then

$$\overline{h_a} = (-iw)^a(-iw)^a \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{iw(x+u)} f(x)\text{g}(\text{u})dudx$$

$$\overline{h_a} = (-iw)^a \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{iwx}} \mathrm{f(x)dx}. (-iw)^a \int_{-\infty}^{\infty} e^{iwu} g(u) du$$

$$\overline{h_a} = \bar{f}_a \overline{g_a} \tag{3.6}$$

Hence convolution theorem proved.

4. Numerical Analysis

In the numerical part, we have developed a digital low-pass filter for denoising the sinusoidal signal. The Sinusoidal function $x = sin\ (2pi600t/F)$ corrupted by mixing high- frequency sinusoidal function into it and considering impulse transfer with Hann window function for denoising the corrupted signal. Then we applied the generalized Fourier trans- form on the corrupted signal and window mixed impulse transfer function, multiplied their generalized Fourier transforms, took the Inverse generalized Fourier transform of the result, and got the denoised signal. It can be seen from Figure-(1) that the filtering by Generalized Fourier transform is better than the frequency domain filtering. From Figure- (2), somebody can observe that when we vary the value of the fractional parameter a, we get a more accurate signal.

5. Conclusion

The definition of the generalized Fourier transform is (3.4), (3.5) shows the convolution of two functions. In (3.6), the Convolution Theorem for the Generalized Fourier Transform is demonstrated. The Low Pass filter is designed using the generalized Fourier transform. The Generalized Fourier transform is superior to the Fourier transform, as the numerical simulation shows. Additionally, it is observed from the Figure-(2) that the filter produces accurate results for various values of a and the and more accurate obtained when the valueof the fractional parameter a is equal to 0.5.

6. Statement and Declarations

Competing Interests:

The authors affirm that the publication of this paper did not involve any conflicts of interest.

Data Accessibility

There is no extra data everything available in this paper.

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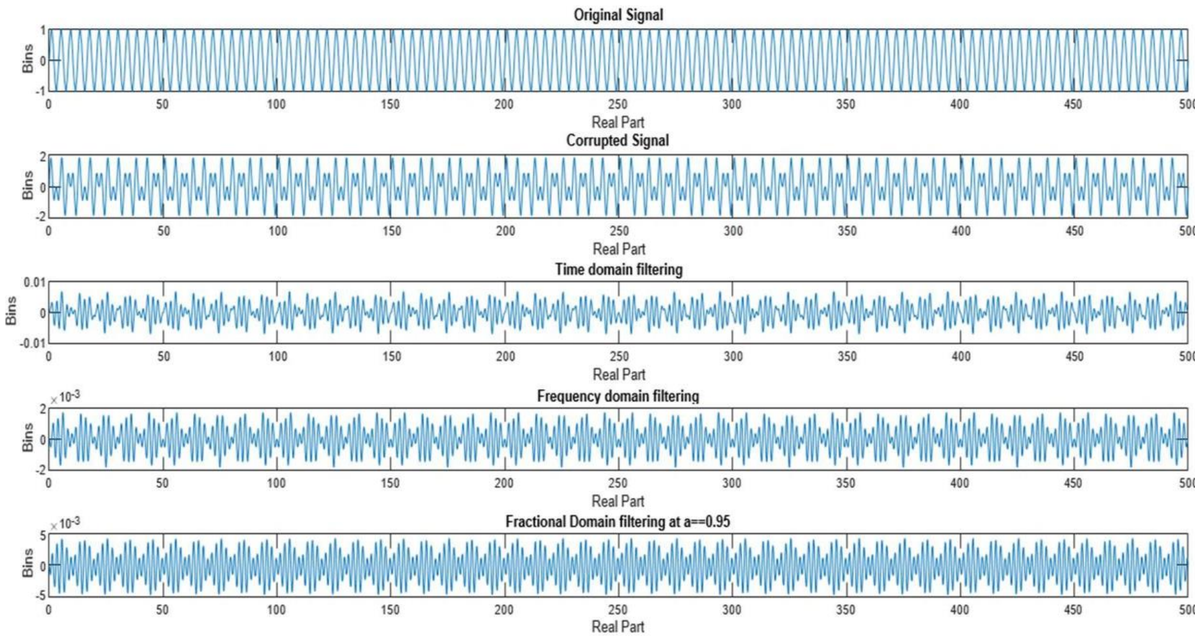

Figure 1: Comparative Graph between Time domain filtering, frequency domain filtering,and Fourier transform in fractional calculus environment filtering at a=0.95.

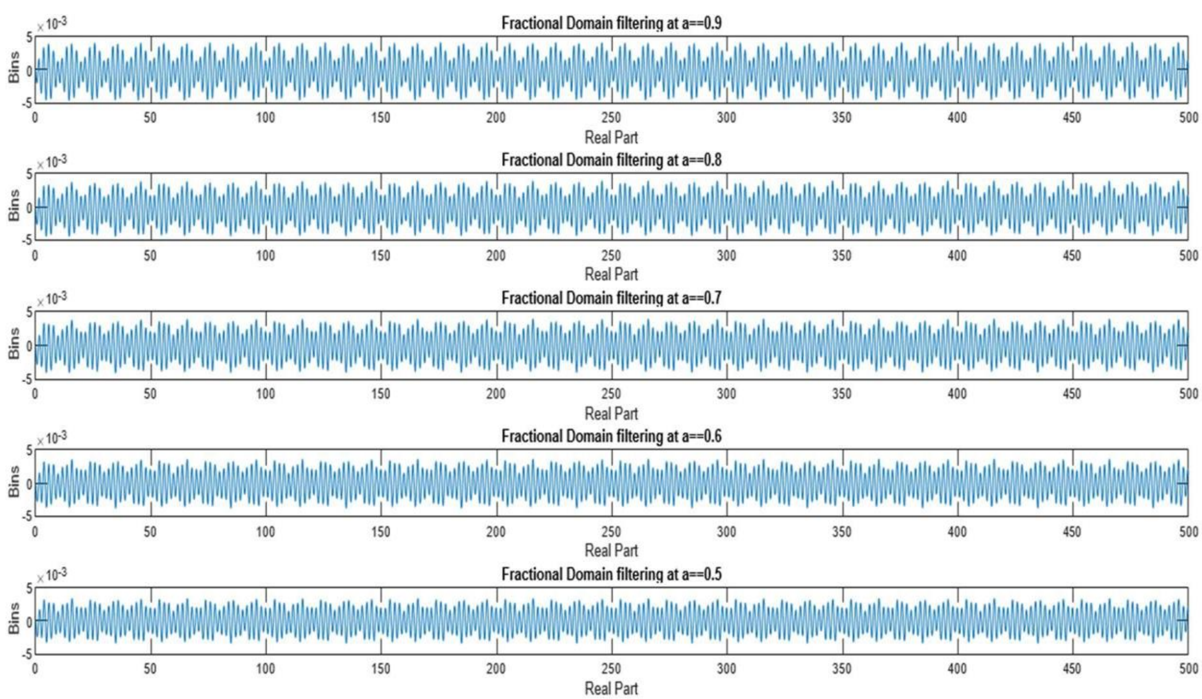

Figure 2: Graph of the filtered signal for various values of fractional parameter a = 0.9, 0.8,0.7, 0.6, 0.5.