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PARTITIONING INTO CONSECUTIVE SQUARE INTEGERS

P. Geetha[1], S. Dilip Jose[2], B. Sridevi[3]
[1]Assistant Professor, Department of Mathematics, Periyar Maniammai Institute of Science & Technology, Periyar Nagar, Vallam, Thanjavur – 613403, maxgeetha@gmail.com

[2]Assistant Professor, Department of Mathematics, Periyar Maniammai Institute of Science & Technology, Periyar Nagar, Vallam, Thanjavur – 613403, dilipjose.rs@pmu.edu

[3]Assistant Professor, Department of Mathematics, Periyar Maniammai Institute of Science & Technology, Periyar Nagar, Vallam, Thanjavur–613403, devisrikarthikeyan2011@gmail.com

Abstract: Seung Kyung Park introduced the r – complete partitions of a positive integer n such that r – complete partition must have 1 as the first part. This paper presents the concept of partitioning into consecutive square integers and an attempt has been given for the theorem based on the partitions of square integers into consecutive terms.
Keywords : Partitions, Complete Partitions, r – Complete Partitions, Integers
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1. Introduction

German mathematician Carl Friedrich Gauss (1777 – 1855) said, "Mathematics is the queen of the sciences – and number theory is the queen of mathematics". Number theory is a branch of pure mathematics devoted primarily to the study of the integers and integer – valued functions. Number theory, an interesting branch of mathematics that deals with integers and their properties plays an important role in discrete mathematics.

The theory of partitions [1,2] has an interesting history. Many other great mathematicians including Cayley, Gauss, Hardy, Lagrange, Littlewood, Rademacher, Schur, Sylvester, in particular Srinivasa Ramanujan have contributed to the development of partition theory. The great Indian mathematician, Srinivasa Ramanujan has left the sign of his brilliance in Number theory throughout his entire life. He has also made some extraordinary contributions to the fields of Hypergeometric series, Elliptic functions, Prime numbers, Bernoulli`s numbers, Divergent series, Continued fractions, Elliptic Modular equations, Highly Composite numbers, Riemann Zeta functions, Partition of numbers, Mock-Theta functions etc.

In fact, it is quite difficult to understand Srinivasa Ramanujan`s mathematics if one does not have the basic foundation in various mathematical subjects. His most outstanding contribution was the formula for $p(n)$, the number of 'partitions' of 'n'. Ramanujan was considered as the master of numbers. He prepared a list of partitions for each of the first 200 integers in 1920. For all these reasons, Ramanujan is hailed as an all time great mathematician like Euler, Gauss or Jacobi for his natural genius. Partition theory [38] has many unique and novel features and challenges both in terms of combinatorial proofs and generating functions. Many of the mathematical sciences have seen applications of partitions recently. Various permutation problems in probability and statistics are intimately linked with Simon Newcomb problem. Nonparametric statistics requires restricted partitions. Particle physics uses partition asymptotics and partition identities. Partition of a convex polygon yields solution to traffic control problems.

Still research is on, in developing the role of various types of partitions like restricted partitions, over partitions and rooted partitions etc., in q – series and combinatorics [3,4 & 5]. Certain special problems in partitions date back to the middle ages. Generally, the combinatorial and formal power series aspects of partitions have found a place in older books on elementary analysis and encyclopaedic surveys of number theory. Considering the applications of partitions in various branches of mathematics and statistics, it is interesting to note the inter play of combinatorial and asymptotic methods.

The magic squares form the nucleus of the theory of partitions [6, 7 & 8] was developed by Srinivasa Ramanujan. His fascination for magic squares led him in his later life to work on this theory. Now consider the partitions of a natural number. Let $p(n)$, denote the partition function $n.$ defined as the number of ways of expressing n as a sum of natural numbers $\leq n.$ For example, 1 has the partition 1; 2 has the partitions 2, 1+1; 3 has the partitions 3, 2+1, 1+1+1, and so on. As n increases, $p(n)$ becomes larger and larger. For example, 6 has the partitions 6, 5+1, 4+2, 4+1+1, 3+3, 3+2+1, 3+1+1+1, 2+2+2, 2+2+1+1, 2+1+1+1+1, 1+1+1+1+1+1.

The following table provides the values of $p(n)$ for $n = 1, 2, \ldots, 20$.

n	1	2	3	4	5	6	7	8	9	10
$p(n)$	1	2	3	5	7	11	15	22	30	42
n	11	12	13	14	15	16	17	18	19	20
$p(n)$	56	77	101	135	176	231	297	385	490	627

Table 1.1

Denoting the number of partitions of n with parts ≤ m by $p_m(n)$, we have the recurrence relation $p_m(n) = p_{m-1}(n) + p_m(n-m)\ (1 < m \leq n)$.

The most famous and broadest of books devoted to partition theory [9, 10, 11 & 12] is "The Theory of Partitions" by G. E. Andrews and the most elementary text is "Integer Partitions" by Andrews and K. Eriksson. The illustration of any positive integer by the way of sums of other positive integers is regarded as the fundamental additive decomposition process.If n is a positive integer, then a partition of n is a non-increasing sequence of positive integers $p_1, p_2, \ldots p_k$ whose sum is n. Each p_i is called a part of the partition. Denoting the number of partitions of the integer n as $p(n)$, $p(5) = 7$. The corresponding partitions are presented below.

$$\begin{aligned} 5 &= 5 \\ &= 4+1 \\ &= 3+2 \\ &= 3+1+1 \\ &= 2+2+1 \\ &= 2+1+1+1 \\ &= 1+1+1+1+1 \end{aligned}$$

We take $p(n) = 0$ for all negative values of n and $p(0)$ is defined to be 1.

Representation of a positive integer as a sum of two or greater squares is also a partition, where each part is a square or square number. MacMahon introduced the concept of a perfect partition of a positive integer n, which is defined as a partition which contains one and only one partition of every lesser number.In 1960, Hoggatt considered sequences such that every positive integer can be represented as a sum of some terms of the sequences and Brown studied such sequences and named complete, which are defined as sequences $(s_1, s_2, \ldots)$ such that every integer can be represented as $\sum_{i=1}^{\infty} \alpha_i s_i$, where $\alpha_i \in S = \{0, 1\}$.There are many kinds of partitions depending on the parts represented. The partition of a positive integer n, can be defined as : A finite non – decreasing sequence $\mu = (\mu_1, \mu_2, \ldots \mu_k)$ such that $\sum_{i=1}^{k} \mu_i = n$ and $\mu_i > 0$ for all $i = 1, 2, \ldots k$. The μ_i are called the parts of the partition and k is called the length of the partition.

2. Survey of Literature

J. J. Sylvester (1857) initiated the study of the partition of numbers. He published a paper "A note on the theory of a point in partitions" and developed partition of an even number into two primes in 1871 and studied further on sub invariants, i.e. semi-invariants to binary quantics of an unlimited order, with rational fractions and partitions in 1882. Hansraj Gupta (1969) presented a historical survey of some aspects of the theory of partitions in his work partitions – A survey. A. K. Agarwal and M. V. Subbaro (1991) presented the some properties of perfect partition function and Combinatorial interpretation of n! [13, 14 & 15]. Seung Kyung Park (1996) contributed the study of complete partitions, recurrence relations and genenating functions of complete partitions. Seung Kyung Park (1997) worked for the study of the enumeration of r – complete partitions and a generalization of complete partitions of a positive integer [16, 17]. Marc Noy (1999) developed enumeration of geometric configurations on a convex Polygon. Alladi (1999) proposed a fundamental invariant in the theory of partitions. Hokyu Lee and Seung Kyung Park (2002) represented the double complete partitions with more specified completeness and worked for the r – subcomplete partitions [18, 19]. Ncvillc Robbins (2002) presented the convolution – type formulas for the number of partitions of n that are not divisible by r, coprime to r in the paper on partition functions and divisor sums [20]. Sylvie Corteel and Jeremy Lovejoy (2003) developed overpartitions. T.C. Brown et al (2003) worked on the partition function of a finite set. James A. Sellers, Andrew V. Sills and Gary L. Mullen (2004) worked for bijections and congruences for generalizations of partition identities of Euler and Guy [21]. James A. Seller (2004) published the results appear which deal with partition functions that exclude specific polygonal numbers as parts [22]. C. S. Srivatsan et al (2006) contributed on gentile statistics and restricted partitions. Hokyu Lee (2006) generalized the perfect partition and found a relation with ordered factorizations [23]. Oystein J. Rodseth (2006) presented the study of enumeration of M – partitions, weak M-partitions and generating functions [24]. Mac Mahon (2006) initiated the study of double perfect partitions and he found a relation with ordered factorizations. Oystein J. Rodseth (2007) produced the some standard results, generating functions and completeness of minimal r – complete partitions [25]. James A. Seller (2007) made significant observations on the parity of the total number of parts in odd – part partitions. Bang Ye Wu and Hsiu - Hui Ou (2007) beautifully discussed on performances of list scheduling for set partition problems. William Y. C. Chen and Kathy Q. Ji (2007) worked on weighted forms of Euler's theorem. Kathy Qing Ji (2008) developed a combinatorial proof of Andrews smallest parts partition function. Michael S. Floater and Tom Lyche (2008) contributed on divided differences of inverse functions and partitions of a convex polygon. Boris Y. Rubinstein (2008) presented expression for restricted partition function through Bernoulli polynomials. Rafael Jakimczuk (2009) contributed restricted partitions elementary methods. Frank G. Garvan and Hamza Yesilyurt worked on shifted and shiftless partition identities which will appear in Int. J. Number Theory [26]. Zeng produced the q-variations of Sylvester's bijection between odd and strict partitions that will appear [27]. Andrews investigations on the number of the smallest parts in the partitions of n were considered which will appear in J. Reine Angew. Math. [28]. George E. Andrews (2009) investigated the partitions with distinct evens and produced companion theorems for distinct evens partitions counted by exceptional parts [29]. Mohamed El Brachraoui (2009) worked the relatively prime partitions with two and three parts for an integer [30]. Ncsrinc Bcnyahia Tani and Sadek Bouroubi (2011) established the enumeration of the partitions of an integer into parts of a specified number of different sizes and especially two sizes [31]. William Y. C. Chen, Ae Ja Yee and Albert J. W. Zhu (2012) contributed the work for Euler's partition theorem with upper bounds on multiplicities [32]. George E Andrews (2013) initiated the work for partitions with early conditions [33]. Shane Chern (2017) investigated the partitions with even parts below odd parts [34]. Jane Y. X. Yang (2018) presented the combinatorial proofs and generalizations on conjectures related with Euler's partition theorem [35].

Motivated by the above, as an amateur number theorist an attempt has been made in the partitioning of consecutive square integers.

3. Preliminaries

Definition 3.1 : A complete partition of an integer n is a partition $\nu = (\nu_1 \nu_2 \dots \nu_k)$ of n, with $\nu_1 = 1$, such that each integer $i, 1 \le i \le n$, can be represented as a sum of elements of $\nu_1 \nu_2 \dots \nu_k$. In other words, each i can be expressed as $\sum_{j=1}^{k} \chi_j \nu_j$, where χ_j is either 0 or 1.

Definition 3.2 : A double complete partition of an integer n is a partition $\nu = (\nu_1^{m_1} \nu_2^{m_2} \dots\dots \nu_l^{m_l})$ of n such that each integer m, with $2 \le m \le n-2$ can be represented by at least two different ways as a sum $\sum_{i=1}^{l} \chi_i \nu_i$ with $\chi_i \in \{0, 1, 2, \dots m_l\}$.

Definition 3.3: For any integer $n \ge 8$, its triple [36 & 37] complete partition of an integer n is a partition $\nu = (\nu_1^{m_1} \nu_2^{m_2} \dots\dots \nu_l^{m_l})$ of n such that each integer m, with $3 \le m \le n-3$ can be represented at least three different ways as a sum $\sum_{i=1}^{l} \chi_i \nu_i$ with $\chi_i \in \{0, 1, 2, \dots m_l\}$.

4. Main results

Definition 4.1 : A consecutive square partitioning of a positive integer n, is a finite non – decreasing sequence $\nu = (\nu_1, \nu_2, \dots, \nu_k) = (1^2, 2^2, 3^2, \dots n^2)$ such that $\sum_{i=1}^{k} \nu_i = n$ and k is the length of the partition.

Theorem 4.2 : Every integer of the form $q = \sum_{s=2}^{n} s^2 + n^2$ which has the base partition $((1^2)^{n^2} 2^2 3^2 \dots\dots n^2)$ may be again partitioned in terms of $1^2, 2^2, 3^2, \dots, n^2$.

Proof : The objective is to exhaust all possible ways of partitioning the given integer in terms of $1^2, 2^2, 3^2, \dots, n^2$. Identify the number of $(n-1)^2$ in n^2. Let it be χ_1. Assign χ_1 as the coefficient of $(n-1)^2$. Now, identify the number of $(n-2)^2$ in $n^2 - \chi_1 (n-1)^2$. Let it be χ_2. χ_2 is taken as the coefficient of $(n-2)^2$. Again identify the number of $(n-2)^2$ in $n^2 - \chi_1(n-1)^2 - \chi_2(n-2)^2$. Let it be χ_3. Continuing this process, if the residue is ultimately χ_0 take the coefficient of 1^2 as χ_0. This process gives the partition of 1^{n^2} which in turn produce the partitions of n. For a clear understanding numerical illustration is presented below.

Numerical Illustration 4.3 :

Partitioning of $q = 79 = 1^{25}\, 4\ 9\ 16\ 25$

Sl. No.	Partitions of n^2		Partitions of q
1.	(3, 22)	(3 4 9 9)	$1^3 4^2 9^3 16 25$

2.	(5, 20)	(5 4 4 4 4 4)	$1^5 4^6 9 16 25$
3.	(7, 18)	(7 9 9)	$1^7 4 9^3 16 25$
4.	(8, 17)	(8 4 4 9)	$1^8 4^3 9^2 16 25$
5.	(9, 16)	(9 16) ; (9 4 4 4 4)	$1^9 4 9 16^2 25$; $1^9 4^5 9 16 25$
6.	(12, 13)	(12 4 9)	$1^{12} 4^2 9^2 16 25$
7.	(16, 9)	(16 9)	$1^{16} 4 9^2 16 25$
8.	(21, 4)	(21 4)	$1^{21} 4^2 9 16 25$

Table 3.1

In 25 there is only one $4^2 = 16$. Therefore coefficient of $4^2 = 1$. Now consider $25 - 16 = 9$. This consists of one 3^2. Therefore, the partition corresponding to 25 is (9 16). Search for 3^2 in 25, there are two 3^2 in 25. Therefore coefficient of 3^2 is taken as 2 and the corresponding partitioning is (9 9). Now, consider $25 - (2 \times 9) = 7$. In 7, search for $2^2 = 4$. There is only one 2^2. So, coefficient of 2^2 is taken as 1. Now, $25 - (9 + 9) - 4 = 3$ which is taken as the coefficient of 1^2. The corresponding partition is $(3, 2^2, 3^2, 3^2)$. In the above partitioning instead of searching for 2^2 in 7, 7 also may be taken as the coefficients of 1^2. Here the corresponding partition is (7, 9, 9). The search ends with the search of 2^2.

This procedure is continued.

Corollary 4.4 : Let $\nu = (\nu_1^{m_1} \nu_2^{m_2} \ldots \nu_l^{m_l})$ be a consecutive partition of a square integer n. Then $\nu_{i+1} \le \sum_{j=1}^{i} k^{j-1} \nu_j$ where ν_{i+1} is the last part of the consecutive square partitioning of an integer.

Proof : In a consecutive square partitioning of an integer n,

$$\nu_{i+1} \le \nu_1^2 + \nu_2^2 + \ldots + \nu_j^2 \le \sum_{s=2}^{n} s^2 + n^2.$$

Conclusion: From the concept of complete partitions of integers an attempt has been given for the partitioning of consecutive square integers. This work may be extended up to partitions of consecutive even integers and to find the generating function for higher order complete partitioning of integers.

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