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# ABSOLUTELY HARMONIOUS LABELING OF RING SUM OF A GRAPH WITH STAR GRAPH

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**Abstract**
Absolutely harmonious labeling$f$ is an injection from the vertex set of a graph $G$ with $q$ edges to the set $\{0,1,2,\ldots,q-1\}$, if each edge $uv$ is assigned $f(u)+f(v)$ then the resulting edge labels can be arranged as $\{a_0, a_1 a_2, \ldots, a_{q-1}\}$ where $a_i = q - i$or $q+i, 0 \leq i \leq q-1$.However, when $G$ is a tree one of the vertex labels may be assigned to exactly two vertices. A graph which admits Absolutely harmonious labeling is called absolutely harmonious graph. In this paper, we study absolutely harmonious labeling of ring sum of a graph with star graph.
**Keywords :** Harmonious labelling, Absolutely harmonious labelling, Star graph, Traingular ladder, Jelly Fish, Jewel graph, $P_n^2$, Globe graph.

## 1. Introduction

In this paper, we consider finite and undirected graphs. A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. A vertex labeling of a graph $G$ is an assignment $f$ of labels to the vertices that induces a label for each edge $xy$ depending on the vertex labels. Seenivasan and Lourdusamy [3] introduced Absolutely harmonious labeling of graphs. In this paper we study the absolutely harmonious labeling of ring sum of a graph with star graph.

**Definition 1.1.**
Absolutely harmonious labeling$f$ is an injection from the vertex set of a graph $G$ with $q$ edges to the set $\{0,1,2,\ldots,q-1\}$, if each edge $uv$ is assigned $f(u)+f(v)$ then the resulting edge labels can be arranged as $\{a_0, a_1, a_2, \ldots, a_{q-1}\}$where $a_i = q-i$ or $q+i, 0 \leq i \leq q-1$. A graph which admits absolutely harmonious labeling is called Absolutely harmonious graph.
**Definition 1.2.**

The Ring sum of two graphs $G_1$ and $G_2$ is a graph consisting of the vertex set$V(G_1) \cup V(G_2)$ and the edges that are either in $G_1$ or $G_2$ but not in both.It isdenoted by $G_1 \bigoplus G_2$.

**Definition 1.3.**

Let $G$ be a graph and u be any vertex of $G$. A new vertex $u'$ is said to be duplication of $u$ if all the vertices which are adjacent to $u$ are adjacent to $u'$.The graph obtained by duplication of $u$ is denoted by $D(G, u')$.

**Definition 1.4.**

The Triangular ladder $TL_n, n \geq 2$ is a graph obtained from the ladder$L_n = P_n \times P_2$ by adding the edges $u_i v_{i+1}$for $1 \leq i \leq n-1$ where $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ are the consecutive vertices of the two copies of the path $P_n$.

## 1. MAIN RESULTS

**Theorem 2.1.**

$P_n^2 \oplus K_{1,n}$is an Absolutely harmonious graph.

**Proof:**Let G=$P_n^2 \bigoplus K_{1,n}$.

The vertex set $V(G) = V_1 \cup V_2$, where $V_1 = V(P_n^2) = \{u_1, u_2, \ldots, u_n\}$ and

$V_2 = V(K_{1,n}) = \{w = u_2, w_1, w_2, \ldots, w_n\}$.

Here, $w_1, w_2, \ldots, w_n$ are the pendant vertices and w is the apex vertex of $K_{1,n}$.

The edge set $E(G) = E_1 \cup E_2$

where$E_1 = E(P_n^2) = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i u_{i+2} : 1 \leq i \leq n-2\}$and $E_2 = E(K_{1,n}) = \{w v_i : 1 \leq i \leq n\}$.

Here, $G$ is of order 2n and size 3n-3.

Now,Define$f: V(G) \rightarrow \{0,1,2,3,\ldots, q-1\}$ as follows:

$$f(u_i) = i-1, 1 \leq i \leq n$$
$$f(w_i) = q-i, 1 \leq i \leq n\,.$$
$$f(w) = f(u_2) = 1$$

Then the induced edge labels are as follows

$$f^*(u_i u_{i+1}) = a_{[q-(2i-1)]}, 1 \leq i \leq n-1$$
$$f^*(u_i u_{i+2}) = a_{[q-2i]}, 1 \leq i \leq n-2$$
$$f^*(w w_i) = a_k; 1 \leq i \leq n; 0 \leq k \leq n-1.$$

From the above $a_0, a_1, a_2, \ldots, a_{q-1}$, where $a_i = q - i\ (or)\ q + i; 0 \leq i \leq q-1$ are the arranged edge labels.

Therefore $f$is an absolutely harmonious labeling of $P_n^2 \oplus K_{1,n}$ and hence$P_n^2 \bigoplus K_{1,n}$ is an Absolutely harmonious graph.

$\blacksquare$

**Example 2.2**

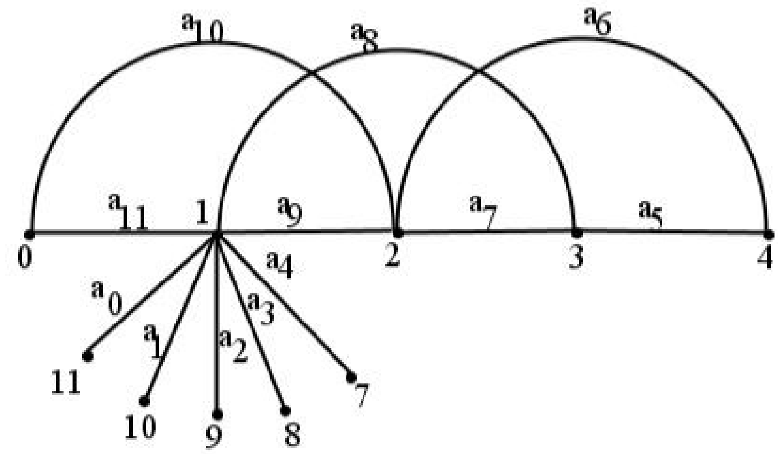


**Figure 1: $P_5^2 \oplus K_{1,5}$**

**Theorem 2.3.**

$J_m \oplus \ K_{1,m}$is an Absolutely harmonious graph.

**Proof:**

Let $G = J_m \oplus \ K_{1,m}$

Let $V(G) = V_1 \cup \ V_2$

where$V_1 = V(J_m) = \{u, v, w, x, y_i : 1 \leq \ i \leq \ m\}$ and

$V_2 = V(K_{1,m}) = \{z = u, z_1, z_2, z_3, \ldots, z_m\}$

Here, $z_1, z_2, z_3, \ldots, z_m$ are the pendant vertices and $z$ is the apex vertex of$K_{1,m}$.

The edge set $E(G) = E_1 \cup \ E_2$where

$E_1 = E(J_m) = \{ux, uw, uv, , xv, vw, xy_i, wy_i : 1 \leq \ i \leq \ m\}$ and

$E_2 = E(K_{1,m}) = \{zz_i : 1 \leq \ i \leq \ m\}$.

Here, G is of order 2m+4 and size 3m+5.

Now, We define $f : V(G) \to \{0,1,2, \ldots, (q-1)\}$ as follows:

$f(u) = f(z) = 0$

$f(v) = 3$

$f(x) = 1$

$f(w) = 2$

$f(y_i) = 2t + 1; \ 1 \leq \ i \leq \ m - 1; 2 \leq \ t \leq \ m$

$f(y_i) = q - 1; i = m$

$f(z_i) = [q - (m + 1)] + j; \ 1 \leq \ i \leq \ m; 0 \leq \ j \leq \ m - 1.$

Then,the edge labels can be clearly arranged as $a_0, a_1, a_2, \ldots \ a_{q-1}$

where$a_i = q - i \ (or) \ q + i; 0 \leq \ i \leq \ q - 1$ .

Therefore, $f$ is an Absolutely harmonious labeling of$J_m \oplus \ K_{1,m}$.

and hence$J_m \oplus \ K_{1,m}$ is an Absolutely harmonious graph.

**Example 2.4**

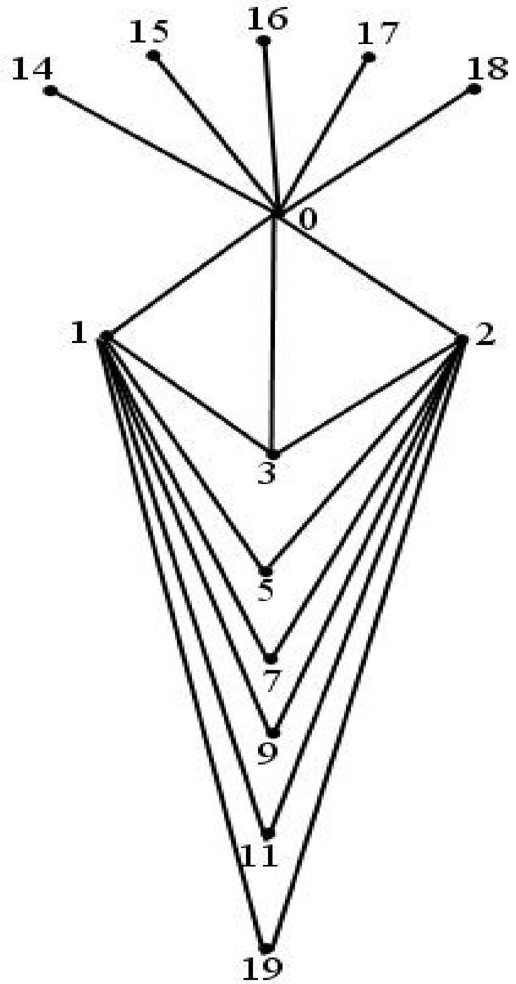


**Figure 2:** $\boldsymbol{J_5 \oplus K_{1,5}}$

**Theorem 2.5**

The Triangular ladder $TL_n \oplus K_{1,n}$ is an Absolutely harmonious graph.

**Proof**

Let $G = TL_n \oplus K_{1,n}$ .

$Let\ V(G) = V_1 \cup V_2$, where$V_1 = V(TL_n) = \{u_i, v_i : 1 \leq i \leq n\}$and

$V_2 = V(K_{1,n}) = \{v_1 = w, w_1, w_2, w_3, \ldots, w_n\}$.

Here, $w_1, w_2, w_3, \ldots, w_m$ are the pendant vertices and w is the apex vertex of $K_{1,n}$.

The edge set $E(G) = E_1 \cup E_2$where

$$E_1 = E(TL_n) = \{u_i u_{i+1}, v_i v_{i+1}, u_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_i : 1 \leq i \leq n\}$$

$E_2 = E(K_{1,n}) = \{w w_i : 1 \leq i \leq n\}$.

Then G is of order 3n and size 5n-3.

Now,Define$f: V(G) \to \{0,1,2,\cdots,q-1\}$ as follows:

$f(v_i) = 2i - 1; 1 \leq i \leq n$

$f(u_1) = 0$

$f(u_i) = 2j; 2 \leq i \leq n; 1 \leq j \leq n-1$
$f(w_i) = q - i; 1 \leq i \leq n$ .
Then the induced edge labels are as follows
$f^*(u_1v_1) = a_{q-1}$
$f^*(u_iv_i) = a_{[q-(4k+1)]}; 2 \leq i \leq n; 1 \leq k \leq n-1$
$f^*(v_iv_{i+1}) = a_{q-4i}, 1 \leq i \leq n-1$
$f^*(u_iv_{i+1}) = a_{[q-(4k+3)]}; 1 \leq i \leq n-1, 0 \leq k \leq n-2$
$f^*(u_1u_2) = a_{q-2}$
$f^*(u_iu_{i+1}) = a_{[q-(3i+k)]}, 2 \leq i \leq n-1, 0 \leq k \leq n-3$
$f^*(v_1w_i) = a_k, 1 \leq i \leq n, 0 \leq k \leq n-1.$
From the above, $a_0, a_1, a_2, \ldots a_{q-1}$
where$a_i = q - i \ (or) \ q + i; 0 \leq i \leq q-1$ are the arranged edge labels.
Therefore $f$ is an absolutely harmonious labeling of Triangular ladder $TL_n \oplus K_{1,n}$and hence the Triangular ladder $TL_n \oplus K_{1,n}$ is an Absolutely harmonious graph.

**Example 2.6**

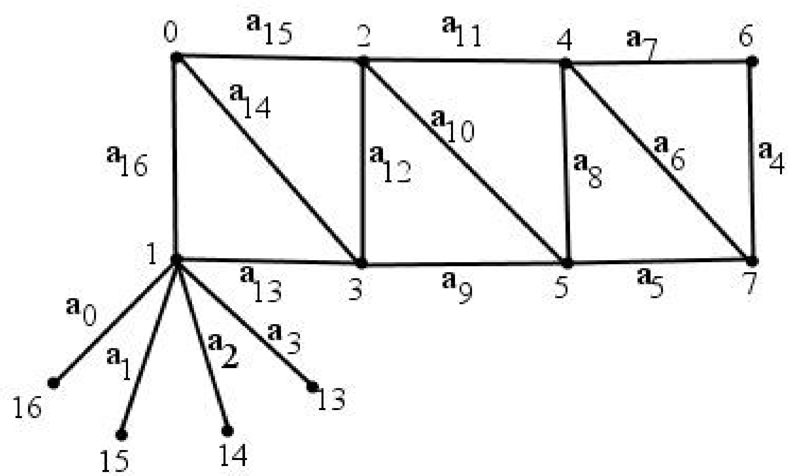


**Figure 3: $TL_4 \oplus K_{1,4}$**

**Theorem 2.7**

The duplication of a vertex in Jellyfish,$D(J(n,n), y')$ is an Absolutely harmonious graph.

**Proof**

Let G=D(J(n,n),$y'$).
The vertex set and the edge set of G are given by
$V(G) = \{(u, v, x, y, y'), (u_i, v_i, 1 \leq i \leq n)\} \ and$
$E(G) = \{[(ux) \cup (uy) \cup (vx) \cup (vy) \cup (xy)] \cup [(uu_i; 1 \leq i \leq n] \cup$
$[(vv_i; 1 \leq i \leq n] \cup [uy'] \cup [xy'] \cup [vy']\}$.
Here,G is of order 2n+5 and size 2n+8.
Now,Define$f: V(G) \to \{0,1,2,3, \ldots, q-1\}$ as follows:
$f(u) = 1$
$f(v) = 2$
$f(x) = 3$

$f(y) = 0$
$f(y') = q - 1$
$f(u_i) = (q-3) - i, 1 \leq i \leq n$
$f(v_i) = 4 + j, 1 \leq i \leq n, 0 \leq j \leq n-1.$
Then the induced edge labels are as follows

$$f^*(uy') = a_0$$
$$f^*(xy') = a_2$$
$$f^*(vy') = a_1$$
$$f^*(uy) = a_{q-1}$$
$$f^*(yv) = a_{q-2}$$
$$f^*(xy) = a_{q-3}$$
$$f^*(xu) = a_{q-4}$$
$$f^*(vx) = a_{q-5}$$
$$f^*(uu_i) = a_{k+3}; 1 \leq i \leq n; 0 \leq k \leq n-1$$
$$f^*(vv_i) = a_{n+k+3};\ n \leq i \leq 1; 0 \leq k \leq n-1.$$

Then,the edge labels can be clearly arranged as $a_0, a_1, a_2, \ldots\ a_{q-1}$where $a_i = q - i\ (or)\ q + i; 0 \leq i \leq q-1$ .

Therefore, f is an absolutely harmonious labeling of the duplication of a vertex in Jellyfish,$D(J(n,n), y')$.

Hence the duplication of a vertex in Jellyfish,$D(J(n,n), y')$ is an Absolutely harmonious graph.

**Example 2.8**

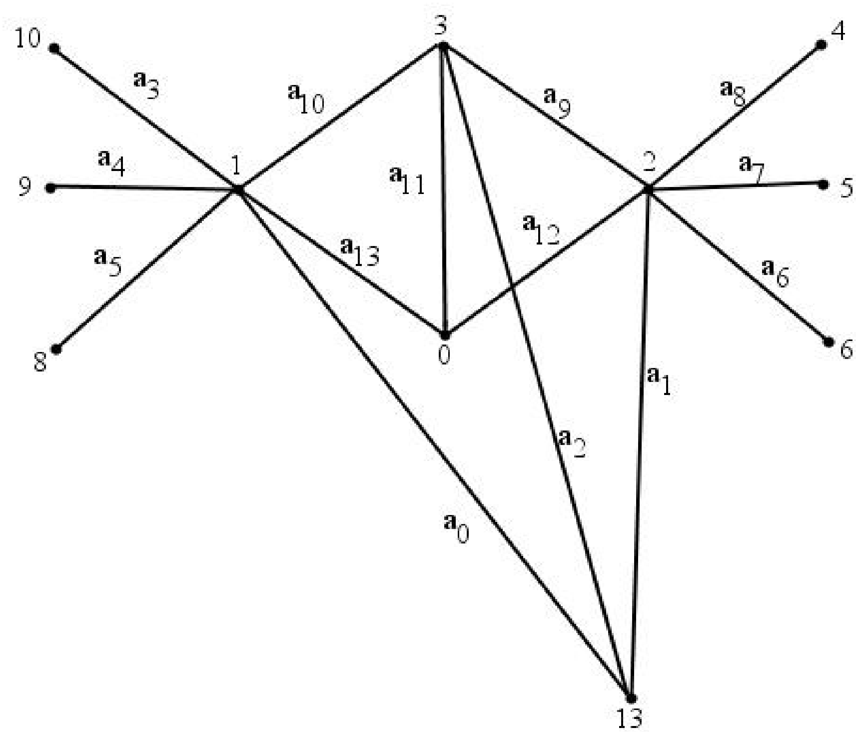


**Figure 4: $D(J(3,3), y')$**

**Theorem 2.9**

The duplication graph $D(TL_n, u')$ is an Absolutely harmonious graph.

**Proof**

Let G be a $D(TL_n, u')$ graph.

Let $V(G) = \{u_i v_i : 1 \le i \le n\} \cup \{u'_1\}$ and

$E(G) = \{u_i u_{i+1}, v_i v_{i+1}, u_i v_{i+1} : 1 \le i \le n-1\} \cup \{u_i v_i : 1 \le i \le n\} \cup \{u'_1 v_1\} \cup \{u'_1 v_2\} \cup \{u'_1 u_2\}$.

Then G is of order 2n+1 and size 4n.

Now, $f: V(G) \to \{0,1,2,3, \ldots, q-1\}$ as follows:

$f(u'_1) = q - 1$

$f(u_1) = 0$

$f(u_i) = 2k; 2 \le i \le n, 1 \le k \le n-1$

$f(v_i) = 2i - 1; 1 \le i \le n$ .

Then the induced edge labels are as follows

$f^*(u'_1 v_1) = a_0$

$f^*(u'_1 u_2) = a_1$

$f^*(u'_1 v_2) = a_2$

$f^*(u_1 v_1) = a_{q-1}$

$f^*(u_i v_i) = a_{[q-(4k+1)]}; 2 \le i \le n; 1 \le k \le n-1$

$f^*(v_i v_{i+1}) = a_{q-4i}, 1 \le i \le n-1$

$f^*(u_i v_{i+1}) = a_{[q-(4k+3)]}; 1 \le i \le n-1, 0 \le k \le n-2$

$f^*(u_1 u_2) = a_{q-2}$

$f^*(u_i u_{i+1}) = a_{[q-(3i+k)]}, 2 \le i \le n-1, 0 \le k \le n-3$

From the above, $a_0, a_1, a_2, \ldots, a_{q-1}$, where $a_i = q - i\ (or)\ q + i; 0 \le i \le q-1$ are the arranged edge labels.

Therefore f is an absolutely harmonious labelingof$D(TL_n, u')$.

Hence $D(TL_n, u')$ is an Absolutely harmonious graph.

**Example 2.10**

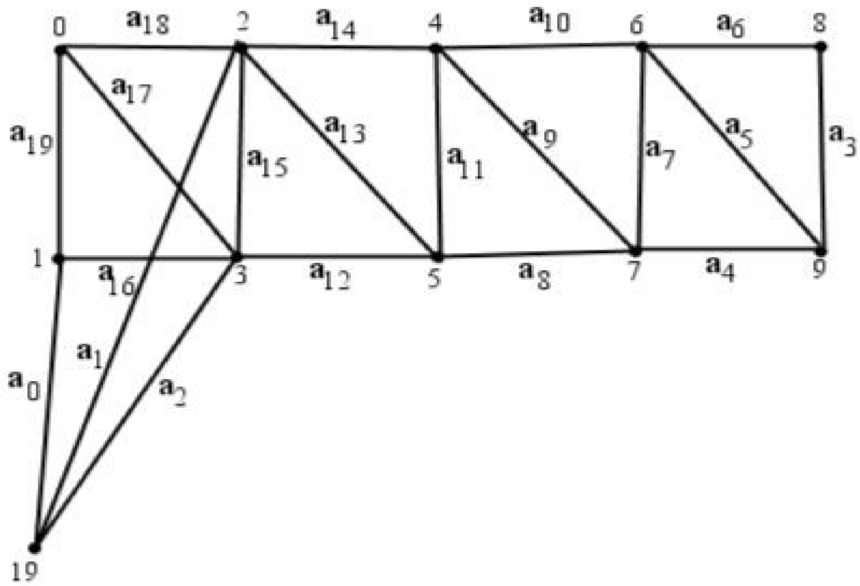


**Figure 5: $D(TL_5, u')$**

**Theorem 2.11**

The graph $(K_2 + m\,K_1) \oplus\ K_{1,n}$is an Absolutely harmonious graph.

**Proof**

Let $G = (K_2 + m\,K_1) \oplus\ K_{1,n}$ .

Let $V(G) = V_1 \cup\ V_2$

where$V_1 = V(K_2 + m\,K_1) = \{x, y, z_1, z_2, \ldots, z_m\}\ and$

$V_2 = V\big(K_{1,m}\big) = \{w = x, w_1, w_2, \ldots, w_m\}$.

The edge set $E(G) = E_1 \cup\ E_2$where

$E_1 = E(K_2 + m\,K_1) = \{xy, xz_i, yz_i : 1 \leq\ i \leq\ m\}$

$and\ E_2 = E\big(K_{1,m}\big) = \{ww_i : 1 \leq\ i \leq\ m\}$.

Here,G is of order 2m+2 and size 3m+1.

Now,Define f:V(G)→ {0,1,2,3,... q-1} as follows

$f(x) = 0$

$f(y) = p - 1$

$f(w_i) = m + i, 1 \leq\ i \leq\ m$

$f(z_i) = i;\ 1 \leq\ i \leq\ m$ .

The induced edge labels are as follows

$f^*(xy) = a_m$

$f^*(xz_i) = a_{q-i}; 1 \leq\ i \leq\ m$

$f^*(yz_i) = a_{m-i}; 1 \leq\ i \leq\ m$

$f^*(xw_i) = a_{q-(m+i)}; 1 \leq\ i \leq\ m.$

From the above, $a_0, a_1, a_2, \ldots, a_{q-1}$ where $a_i = q - i\ (or)\ q + i; 0 \leq\ i \leq\ q - 1$ are the arranged edge labels.

Therefore, f is an absolutely harmonious labeling

and hence$(K_2 + m\,K_1) \oplus\ K_{1,n}$ is an Absolutely harmonious graph.

**Example 2.12**

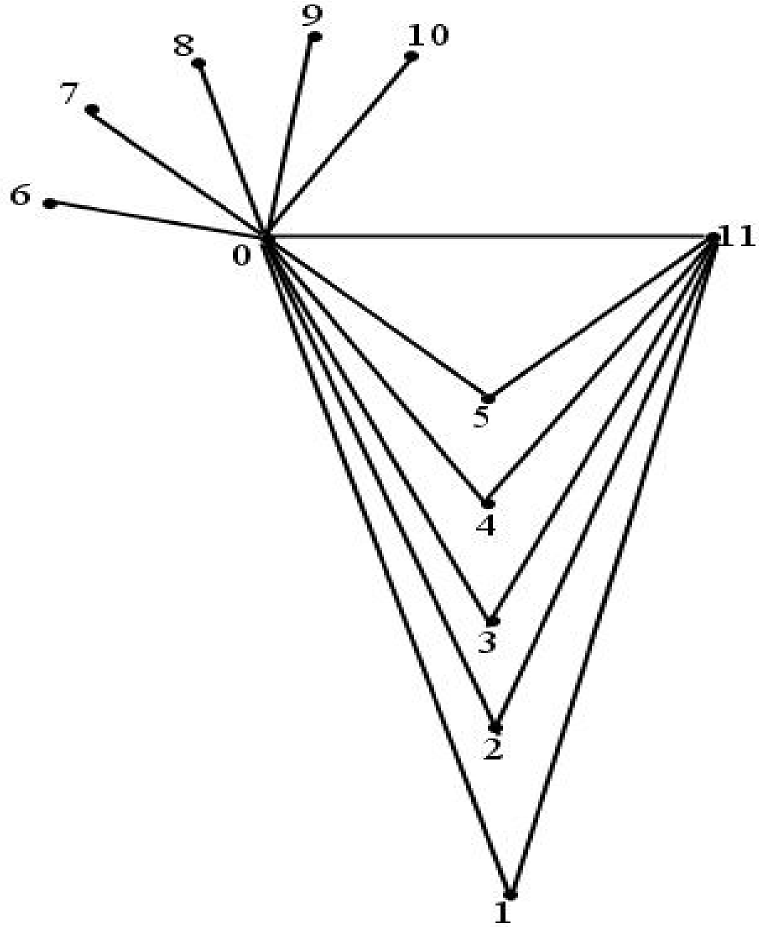


Figure 6: $(K_2 + 5K_1) \oplus K_{1,5}$

3. **References**

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