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NON-COMMUTING GRAPHS ON DIHEDRAL GROUPS

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Abstract

Let G be a non-abelian group and $\Omega \subset G$. The *Non-Commuting graph* $\Gamma = (G, \Omega)$, has Ω as its vertex set with two distinct elements ofΩ joined by an edge when they do not commute in G. In this article, we investigate among some properties of Non-Commuting graphs and the degree of all vertices in Γ. We also study a necessary and sufficient condition for Γ to be Eulerian.

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1 INTRODUCTION

The study of algebraic structures, using the properties of graphs, becomes an exciting research topic in the last twenty years, leading to many fascinatingresults and questions. For example, the study zero-divisor graphs, total graph of commutative rings and commuting graph of groups has attracted many researchers towards this dimension. One can refer [2, 3] for such studies. Theconcept of non-commuting graph has been studied in [1], where as the concept of commuting graph has been found in [4]. For basic defns one can refer [5, 6, 7, 9]. Before starting let us introduce some necessary notation and definitions.

Let G be a group. The center of a group G is denoted by $Z(G)$. Let Ω be any nonempty subset of G. The centralizer of Ω in G is the set of elements of G which commutes with every element of Ω and it is denoted by $C_\Omega(G)$. Here we consider the following way: Take $G \setminus Z(G)$ as the vertices of G and join two distinct vertices x and y whenever x and y do not commute with each others. Note that if G is abelian, then Γ is the null graph. For any integer $n \geq 3$, theDihedral group $2n$ is given by $D_{2n} = \langle r, s : s^2 = r^n = 1, rs = sr^{-1}\rangle$. In this article, we consider the Non-Commuting graphs in the context of dihedral group D_{2n}. For any subset Ω of D_{2n}, the *Non-Commuting graph*

$\Gamma = (G, \Omega)$ has Ω as its vertex set $G \setminus Z(G)$ with two distinct vertices in Ω are adjacent if they do not commute with each other in D_{2n}.

We consider simple connected undirected graphs, with no loops or multiple edges. For any graph Γ, we denote the sets of the vertices and the edges of by $V(\Gamma)$ and $E(\Gamma)$, respectively. The degree $deg_\Gamma(v)$ of a vertex v in Γ is the number of edges incident to v and if the graph is understood, then we denote$d\{\Gamma\}(v)$ simply by deg_Γ. The order of Γ is defined $|V(\Gamma)|$ and its maximum and its minimum degrees will be denoted, respectively, by $\Delta(\Gamma)$ and $\delta(\Gamma)$. Agraph Γ is regular if the degrees of all vertices of Γ are the same. A subset X ofthe vertices of Γ is called a clique if the induced subgraph on X is a complete graph. The maximum size of a clique in a graph Γ is called the clique number of Γ and denoted by $\omega(\Gamma)$.

A path P is a sequence $v_0e_1v_1e_2 \ldots e_kv_k$ whose terms are alternately distinct vertices and distinct edges, such that for any i, $1 \leq i \leq k$, the ends of e_i arev_{i1} and v_i. In this case P is called a path between v_0 and v_k. The number $k$is called the length of P. If v_0 and v_k are adjacent in Γ by an edge e_{k+1}, then$P \cup \{e_{k+1}\}$ is called a cycle. The length of a cycle defined the number of its edges. The length of the shortest cycle in a graph Γ is called girth of Γ and denoted by girth (Γ). A Hamilton cycle of Γ is a cycle that contains every vertex of Γ. If v and w are vertices in Γ, then $d(v, w)$ denotes the length ofthe shortest path between v and w. The largest distance between all pairs of the vertices of Γ is called the diameter of Γ, and is denoted by $diam(\Gamma)$. A graph Γ is connected if there is a path between each pair of the vertices of Γ. A planar graph is a graph that can be embedded in the plane so that no two edges intersect geometrically except at a vertex which both are incident.

It is well known that any compact surface is either homeomorphic to a sphere, or to a connected sum of g tori, or to a connected sum of k projectiveplanes (see [8], Theorem 5.1). We denote by S_g the surface formed by a connected sum of g tori. The number g is called the genus of the surface S_g. Also a graph Γ is called planar if $\gamma(G) = 0$, and it is called toroidal if $\gamma(G) = 1$.Note that, a graph G is perfect if neither G nor $\overline{G}$ contains any induced odd cycle of degree at least five.

In Section 2 of the paper, we study some graph properties of the non- commuting graph Γ of D_{2n}. We see that Γ is always connected, its diame- ter,perfect matching, number of triangles and number of C_4. We also study a necessary and sufficient condition for Γ to be Eulerian.

Lemma 1.1. *[8]* The following statements hold:

1. $\gamma(K_n) = \lceil \frac{1}{12}(n-3)(n-4) \rceil$ if $n \geq 3$;
2. $\gamma(K_{m,n}) = \lceil \frac{1}{4}(m-2)(n-2) \rceil$ if $m, n \geq 2$.

Note that Kuratowski's Theorem [[10],Theorem 6.2.2] says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.

Theorem 1.2. *[10] A graph G is a claw-free graph if G does not contain a $K_{1,3}$ as an induced subgraph.*

2 Main Results

Throughout this section, $n \geq 3$ is an integer and $D_{2n} =< r,\ s\ :\ s^2 = r^n = 1, rs = sr^{-1} >$. Let Γ be the simple undirected graph with vertex set $D_{2n} \backslash Z(D_{2n})$ in which two distinct vertices are adjacent if and only if they do not commute.

In the first lemma, we obtain the degree of all the vertices the graph Γ.

Lemma 2.1. *Let $n \geq 3$ be an integer.*
(i). For $1 \leq i \neq \frac{n}{2} \leq n$ when n is even and for $1 \leq i \leq n$ when n is odd, then $deg_\Gamma(r^i) = n$;
(ii). For $1 \leq i \leq n$, then $deg_\Gamma(sr^i) = \begin{cases} 2n-4 & \text{if } n \text{ is even;} \\ 2n-3 & \text{if } n \text{ is odd.} \end{cases}$

Proof. (i) Let $x = r^i$ for some i, $1 \leq i \neq \frac{n}{2} \leq n$ when n is even or for i, $1 \leq i \leq n$ when n is odd. Then x is only adjacent to every element from $\{s, sr, sr^2, \ldots, sr^{n-1}\}$ in Γ and so $\deg_\Gamma(x) = n$.

(ii) Suppose n is even. Let $V_1 = \{r, r^2, \ldots, r^{\frac{n-1}{2}}, r^{\frac{n+1}{2}}, \ldots, r^{n-1}\}$ and $V_2 = \{s, sr, sr^2, \ldots, sr^{n-1}\}$. Then $V(\Gamma) = V_1 \cup V_2$ and $|V(\Gamma)| = 2n - 2$.
Let $x = sr^i$ for $1 \leq i \leq n$. Then x is adjacent to every other vertex from $V(\Gamma) \setminus \{sr^j\}$ in Γ where if $j > i$, then $j - i = \frac{n}{2}$ $(i > j, i - j = \frac{n}{2})$ and so $\deg_\Gamma(x) = 2n - 4$.
Suppose n is odd and $x = sr^i$ for $1 \leq i \leq n$. Then x is adjacent to every other vertex in Γ and so $\deg_\Gamma(x) = 2n - 3$. □

One can have the following corollary from the above lemma.

Corollary 2.2. *Let $n \geq 3$ be an integer. For $1 \leq i \leq n$, then*

1. *$diam(\Gamma) = 2$;*
2. *$gr(\Gamma) = 3$;*
3. *Γ is connected.*

In the following theorem, we find the number of edges in the graph Γ.

Theorem 2.3. *Let $n \geq 3$ be an integer. Then the number of edges*
$$\epsilon(\Gamma) = \begin{cases} \frac{3n(n-2)}{2} & \text{if } n \text{ is even;} \\ \frac{3n(n-1)}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. **Case (i).** Let n be even. Let $V_1 = \{r, r^2, \ldots, r^{\frac{n-1}{2}}, r^{\frac{n+1}{2}}, \ldots, r^{n-1}\}$ and $V_2 = \{s, sr, sr^2, \ldots, sr^{n-1}\}$. Let $x \in V_1$. Then x adjacent to every element to V_2 in Γ. This gives Γ contains $(n-2)n$ edges. Every element sr^i in V_2 is adjacent to every element from $V_2 \setminus \{sr^i, sr^j\}$, where if $j > i$, then $j - i = \frac{n}{2}$ $(i > j, i - j = \frac{n}{2})$. This yields Γ contains $\frac{n(n-1)}{2} - \frac{n}{2}$ edges. Since $< V_1 > \cong \overline{K_{n-2}}$ in Γ, Γ contains only $n(n-2) + \frac{n(n-1)}{2} - \frac{n}{2} = \frac{3n(n-2)}{2}$ edges.
Case (ii). Let n be odd. Let $V_1 = \{r, r^2, \ldots, r^{n-1}\}$ and $V_2 = \{s, sr, sr^2, \ldots, sr^{n-1}\}$. Every element of $x \in V_1$ is adjacent to every element from V_2 in Γ. This gives Γ contains $n(n-1)$ edges. Every element in V_2 is adjacent to every other element from V_2 in Γ. This implies Γ contains $\frac{n(n-1)}{2}$ edges. Since $< V_1 > \cong \overline{K_{n-1}}$ in Γ, Γ contains only $n(n-1) + \frac{n(n-1)}{2} = \frac{3n(n-1)}{2}$ edges. □

In the following theorem, we find the number of triangles in the graph Γ.

Theorem 2.4. *Let $n \geq 3$ be an integer. Then the number of triangles in*
$$\Gamma = \begin{cases} \frac{n(n-2)(13n-28)}{24} & \text{if } n \text{ is even;} \\ \frac{n(n-1)(4n-5)}{6} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Suppose n is even. Let $V_1 = \{r, r^2, \ldots, r^{\frac{n-1}{2}}, r^{\frac{n+1}{2}}, \ldots, r^{n-1}\}$ and $V_2 = \{s, sr, sr^2, \ldots, sr^{n-1}\}$. Let $H_1 = \{s, sr, sr^2, \ldots, sr^{\frac{n}{2}-1)}\}$ and $H_2 = \{sr^n, sr^{\frac{n}{2}+1}, \ldots, sr^{n-1}\}$. Then $H_1 \cup H_2 = V_2$ and $|H_1| = |H_2| = \frac{n}{2}$. Also $< H_1 >=< H_2 >= K_{\frac{n}{2}}$ in Γ.

Case (i). $< \{x, y, z\} >\cong K_3$ in Γ with $x \in V_1$.

Sub case (a). $< \{x, y, z\} >\cong K_3$ in Γ with $x \in V_1$ and either $y, z \in H_1$ or $y, z \in H_2$.

If $x \in V_1$ and $y, z \in H_1$, then Γ contains $(n-2)\frac{n}{2}(\frac{n}{2}-1)/2 = \frac{n(n-2)^2}{8}$ triangles. If $x \in V_1$ and $y, z \in H_2$. This gives Γ contains $(n-2)\frac{n}{2}(\frac{n}{2}-1)/2 = \frac{n(n-2)^2}{8}$ triangles. In this case, Γ contains $\frac{n(n-2)^2}{4}$ triangles.

Sub case (b). $< \{x, y, z\} >\cong K_3$ in Γ with $x \in V_1$, $y \in H_1$ and $z \in H_2 \setminus \{w\}$ where z and w are adjacent in Γ. Note that every element $y \in H_1$ is not adjacent to exactly one element $w \in H_2$ in Γ. This implies Γ contains $(n-2)(\frac{n}{2}-1)\frac{n}{2} = \frac{n(n-2)^2}{4}$. Since $< V_1 >\cong \overline{K_{n-2}}$ in Γ, Γ contains exactly $\frac{n(n-2)^2}{4}$ triangles. From sub cases (a) and b, Γ contains $\frac{n(n-2)^2}{2}$ triangles.

Case (ii). $< \{x, y, z\} >\cong K_3$ in Γ with $x, y, z \in H_1$ or $x, y, z \in H_2$. Since $|H_1| = |H_2| = \frac{n}{2}$, Γ contains $\frac{n(n-2)(n-4)}{24}$ triangles.

From cases (i) and (ii), Γ contains $\frac{n(n-2)^2}{2} + \frac{n(n-2)(n-4)}{24} = \frac{n(n-2)(13n-28)}{24}$ triangles.

Suppose n is odd. Let $V_1 = \{r, r^2, \ldots, r^{n-1}\}$ and $V_2 = \{s, sr, sr^2, \ldots, sr^{n-1}\}$. Every element of $x \in V_1$ is adjacent to every element from V_2 in Γ. If $< \{x, y, z\} >\cong K_3$ in Γ, then (i) $x \in V_1$ and $y, z \in V_2$ or (ii) $x, y, z \in V_2$. This gives Γ contains $(n-1)n\frac{(n-1)}{2} + \frac{n(n-1)(n-2)}{6} = \frac{n(n-1)(4n-5)}{6}$ triangles. □

In the following theorem, we find the number of induced subgraph C_4 in the graph Γ.

Theorem 2.5. *Let $n \geq 3$ be an integer and n be even. Then Γ contains exactly $\frac{n(n-2)(n-3)}{4}$ number of induced subgraph C_4.*

Proof. As given in Theorem 2.4, consider the subsets V_1, V_2, H_1 and H_2. Note that every $x = sr^i \in H_1$ $(1 \leq i \leq \frac{n}{2}-1)$ is not adjacent to only one element $x = sr^j \in H_2$ in Γ with $sr^i sr^j = sr^j sr^i$. Then Γ contains an induced subgraph C_4 by the vertices x, y, sr^i, sr^j for every $x, y \in V_1$, $sr^i \in H_1$ and $sr^j \in H_2$. Since $|V_1| = n-2$, the number of C_4 in Γ is $\frac{(n-1)(n-3)}{2}$ and also $|H_1| = |H_2| = \frac{n}{2}$. Hence, the number of induced subgraph C_4 in Γ is exactly $\frac{n(n-2)(n-3)}{4}$. □

In the following theorem, we obtain a necessary and sufficient condition for the graph Γ to be Eulerian.

Theorem 2.6. *Let $n \geq 3$ be an integer. Then Γ is Eulerian if and only if n is even.*

Proof. Proof follows from Lemma 2.1 □

In the following theorem, we discuss the Hamiltonian nature of Γ.

Theorem 2.7. *Let $n \geq 3$ be an integer. Then,*

1. *Γ is vertex pancyclic;*
2. *Γ is Hamiltonian.*

Proof. (1) Suppose n is even. As given in Theorem 2.4, consider the subsets V_1, V_2, H_1 and H_2. Let $x \in V(\Gamma)$.
For $m = 3$. Since $|H_1| = |H_2| \geq 2$, every element in H_1 is adjacent to at least one element from H_2 in Γ. If $x \in V_1$, then choose $y, z \in H_1 \cup H_2$ such that y and z are adjacent in Γ. This gives a cycle of order 3 containing the vertex x. If $x \in H_1 (\, x \in H_2)$, then choose $y \in H_2$ $(x \in H_1)$ with x and y are adjacent in Γ and $z \in V_1$. This gives a cycle of order 3 containing the vertex x.

For $m \geq 4$. Consider a cycle $C_1 : s - sr^{\frac{n}{2}-1} - sr^{\frac{n}{2}} - sr^{n-1} - s$ in Γ of order

For $m \geq 4$. Consider a cycle $C_1 : s - sr^{\frac{n}{2}-1} - sr^{\frac{n}{2}} - sr^{n-1} - s$ in Γ of order 4. Since $< H_1 >$ is a complete subgraph in Γ, adding one by one element from $H_1 \setminus \{s, sr^{\frac{n}{2}-1}\}$ in the order of $sr, sr^2, \ldots, sr^{\frac{n}{2}-2}$ in between the vertices s and $sr^{\frac{n}{2}-1}$ in the cycle C_1, we have a spanning cycle in Γ of order m where $4 \leq m \leq \frac{n}{2} + 2$ Since $< H_2 >$ is a complete subgraph in Γ, adding one by one element from $H_2 \setminus \{sr^{\frac{n}{2}}, sr^{n-1}\}$ in the order of $sr^{\frac{n}{2}+1}, \ldots, sr^{n-2}$ in between the vertices $sr^{\frac{n}{2}}$ and sr^{n-1} in the cycle C_1, we have a spanning cycle C_1 in $< H_1 \cup H_2 >$ of order m where $4 \leq m \leq n$.

Adding r^i for $1 \leq i \neq \frac{n}{2} \leq n-1$ in between the vertices sr^{i-1} and sr^i in the cycle C_1, we have a cycle in Γ of order m where $4 \leq m \leq 2n-2$. Thus, we have a cycle of order m where $4 \leq m \leq 2n-2$ containing an element from $\{s, sr^{\frac{n}{2}-1}, sr^{\frac{n}{2}}, sr^{n-1}\}$ in Γ. As the same above argument, one can have a cycle of order m where $4 \leq m \leq 2n-2$ containing any element from $H_1 \cup H_2$ in Γ.

Let $x \in V_1$. For $m = 4$, consider the cycle $x - s - y - sr^{n-1} - x$ in Γ of order 4 where $y \in V_1 \setminus \{x\}$. Note that $x = r^i$ for $1 \leq i \neq \frac{n}{2} \leq n-1$. For $m \geq 5$. Consider cycle $C_2 :\ r^i - s - sr^{\frac{n}{2}-1} - sr^{\frac{n}{2}} - sr^{n-1} - r^i$ in Γ of order 5. Proceeding the same above argument adding the elements in the order of $sr, sr^2, \ldots, sr^{\frac{n}{2}-2}, sr^{\frac{n}{2}+1}, \ldots, sr^{n-2}, r, \ldots, r^{i-1}, r^{i+1}, \ldots, r^{n-1}$ in C_2. Hence for each vertex x in Γ we have a cycle of order m where $3 \leq m \leq 2n-2$ containing x and so Γ is vertex pancyclic. Suppose n is odd. One can easily prove as the same above argument.

(2) Proof follows from above (1). □

In the following theorem, we obtain a necessary and sufficient condition for the graph Γ to be chordal.

Theorem 2.8. *Let $n \geq 3$ be an integer. Then Γ is chordal if and only if n is odd.*

Proof. 1. Suppose n is odd.
Let $V_1 = \{r, r^2, \ldots, r^{n-1}\}$ and $V_2 = \{s, sr, sr^2, \ldots, sr^{n-1}\}$. Then $V_1 \cup V_2 = V(\Gamma)$. Note that $< V_2 >= K_n$. Let $S \subseteq V(\Gamma)$ with $|S| \geq 4$. If $S \subseteq V_1$, then by $< V_1 >= \overline{K_{n-1}}$ we have $< S >$ is not an induced cycle subgraph of order $|S| \geq 4$. If $S \subseteq V_2$, then by $< V_2 >= K_n$ we have $< S >$ is not an induced cycle subgraph of order $|S| \geq 4$. Suppose $S \cap V_1 \neq \phi$ and $S \cap V_2 \neq \phi$. If $|S \cap V_2| \geq 3$, then $< S >$ contains a subgraph K_3 and so $< S >$ is not an induced cycle subgraph of order $|S| \geq 4$. If $|S \cap V_2| = 2$, then $|S \cap V_1| \geq 2$ and $< S >$ contains a subgraph K_3. This gives that $< S >$ is not an induced cycle subgraph of order $|S| \geq 4$. Hence Γ is chordal if n is odd.

Conversely assume that Γ is chodal. Suppose n is even. By Theorem 2.3, Γ has at least one induced cycle subgraph of order 4, a contradiction. Hence Γ is chordal if one and only if n is odd. □

In the following theorem, we obtain a necessary and sufficient condition for the graph Γ to be claw-free.

Theorem 2.9. *Let $n \geq 3$ be an integer. Then,*

1. *Γ is claw-free if and only if $G = D_6$ or $G = D_8$;*
2. *Γ is not unicyclic.*

Proof. (1) Suppose $G = D_6$ or $G = D_8$. Let $S \subseteq V(\Gamma)$ with $|S| = 4$. Since $< S >= K_4$ or $< S >$ contains a cyclic subgraph of order 4 in Γ, we have $deg_\Gamma(x)$ is at least two for every $x \in S$ and so $< S > \neq K_{1,3}$ in Γ. Hence Γ is claw-free.

Conversely assume that Γ is claw-free. Suppose $n \geq 10$. Let $V_1 = \{r, r^2, \ldots, r^{\frac{n-1}{2}}, r^{\frac{n+1}{2}}, \ldots, r^{n-1}\}$ and $V_2 = \{s, sr, sr^2, \ldots, sr^{n-1}\}$. Then $|V_1| \geq 4$ and $< V_1 > \cong \overline{K_{|V_1|}}$ in Γ. Note that every element in V_2 is adjacent to every element from V_1 in Γ. This implies that Γ contains an induced subgraph $K_{1,3}$ by the elements $\{a, b, c, d\}$ where $a, b, c \in V_1$ and $d \in V_2$. Hence Γ is claw-free if and only if $G = D_6$ or $G = D_8$.

(2) Proof follows from Theorem 2.3. □

In the following theorem, we obtain the clique number of the graph Γ.

Theorem 2.10. *Let $n \geq 3$ be any integer. Then*
$$\omega(\Gamma) = \begin{cases} \frac{n}{2}+1 & \text{if } n \text{ is even} \\ n+1 & \text{if } n \text{ is odd} \end{cases}$$

Proof. Suppose n is even. As given in Theorem 2.4, consider the subsets $V_1,\ V_2,\ H_1$ and H_2. Since $< H_1 >=< H_2 >= K_{\frac{n}{2}}$ in Γ, $\omega(\Gamma) \geq \frac{n}{2}$. Since every element in V_1 is adjacent to every element to V_2 in Γ and $|V_1| \geq 2$, we have $\omega(\Gamma) \geq \frac{n}{2}+1$. Note that $< V_1 >\cong \overline{K_{n-2}}$ and every element in H_1 is not adjacent to exactly one element to H_2 in Γ and vice versa. If S is a maximal complete subgraph in Γ, then $H_1 \subseteq S$ or $H_2 \subseteq S$ and $|S \cap V_1| = 1$. This turns that $\omega(\Gamma) = \frac{n}{2}+1$.

Suppose n is odd. Let $V_1 = \{r, r^2, \ldots, r^{n-1}\}$ and $V_2 = \{s, sr, sr^2, \ldots, sr^{n-1}\}$. Since $< V_2 >= K_n$ in Γ, $\omega(\Gamma) \geq n$. Every element of $x \in V_1$ is adjacent to every element from V_2 in Γ. Then $\omega(\Gamma) \geq n+1$. Since $< V_1 >\cong \overline{K_{n-2}}$, $\omega(\Gamma) = n+1$. □

In the following theorem, we obtain the chromatic number of the graph Γ.

Theorem 2.11. *Let $n \geq 3$ be any integer. Then*
$$\chi(\Gamma) = \begin{cases} \frac{n}{2}+1 & \text{if } n \text{ is even} \\ n+1 & \text{if } n \text{ is odd} \end{cases}$$

Proof. Suppose n is even. As given in Theorem 2.4, consider the subsets $V_1,\ V_2,\ H_1$ and H_2. By Theorem 2.10, $\omega(\Gamma) = \frac{n}{2}+1$ and so $\chi(\Gamma) \geq \frac{n}{2}+1$. It is enough to show that $\chi(\Gamma) \leq \frac{n}{2}+1$. Every element in H_1 is not adjacent to exactly one element to H_2 in Γ and vice versa. Assign one color to this pair of vertices in $< V_2 >$. Since $< H_1 >= K_{\frac{n}{2}}$ in Γ we have $\chi(< V_2 >) = \frac{n}{2}$ in Γ. Since every element in V_1 is adjacent to every element to V_2 in Γ and $|V_1| \geq 2$, we have $\omega(\Gamma) \geq \frac{n}{2}+1$. Since $< V_1 >\cong \overline{K_{n-2}}$ and assigning one color to all vertices of $< V_1 >$ in Γ, we have $\omega(\Gamma) \leq \frac{n}{2}+1$. Hence $\omega(\Gamma) = \frac{n}{2}+1$.

Suppose n is odd. Let $V_1 = \{r, r^2, \ldots, r^{n-1}\}$ and $V_2 = \{s, sr, sr^2, \ldots, sr^{n-1}\}$. By Theorem 2.10, $\omega(\Gamma) = n+1$ and so $\chi(\Gamma) \geq n+1$. It is enough to show that $\chi(\Gamma) \leq n+1$. Note that $< V_2 >= K_n$ in Γ and every element of $x \in V_1$ is adjacent to every element from V_2 in Γ. Since $|V_1| \geq 2$, we have $\chi(\Gamma) \geq n+1$. Since $< V_1 >\cong \overline{K_{n-2}}$ and sssigning one color to all vertices of $< V_1 >$ in Γ, we have $\chi(\Gamma) \leq n+1$. Hence $\chi(\Gamma) = n+1$. □

In the following corollary, we discuss the nature of weakly perfect of Γ.

Corollary 2.12. *Let $n \geq 3$ be any integer. Then Γ is weakly perfect.*

Proof. Proof follows from Theorem 2.10 and Theorem 2.11. □

In the following corollary, we discuss the nature of perfectness of Γ.

Theorem 2.13. *Let $n \geq 3$ be an integer. Then Γ is a perfect graph.*

Proof. Suppose n is even. As given in Theorem 2.4, consider the subsets V_1, V_2, H_1 and H_2. Let $S \subseteq V(\Gamma)$ with $|S|$ is an odd order greater than or equal to 5. Suppose $< S >$ is an induced cycle subgraph of Γ. Suppose $S \subseteq V_1$. Then $< S >$ is a totally disconnected subgraph in Γ and so $|S \cap V_2| \geq 1$. Suppose $|S \cap V_1| \geq 2$. Every element of $x \in V_1$ is adjacent to every element from V_2 in Γ. This implies $< S >$ contains at least one vertex whose degree is $|S| - 2 \geq 3$. This imples $|S \cap V_1| \leq 1$. If $|S \cap V_1| = 1$, $< S >$ contains one vertex whose degree is $|S| - 1 \geq 4$. Thus $S \subseteq V_2$. Let $x \in V_2$. Then $\deg_{<S>}(x)$ is either $|S| - 1$ or $|S| - 2$ and so $< S >$ contains at least one vertex whose degree is $|S| - 2 \geq 3$, a contradiction. Hence Γ contains no induced cycle subgraph of odd order greater than or equal to 5.

Suppose n is odd. Let $V_1 = \{r, r^2, \ldots, r^{n-1}\}$ and $V_2 = \{s, sr, sr^2, \ldots, sr^{n-1}\}$. Note that $< V_1 > \cong \overline{K_{n-1}}$ and $< V_2 >= K_n$ in Γ. Then either $< S >$ contains at least one vertex whose degree is $|S| - 2 \geq 3$ or $< S >$ is disconnected in Γ. Hence Γ contains no induced cycle subgraph of odd order greater than 5.

For $\overline{\Gamma}$. Suppose n is even. As given in Theorem 2.4, consider the subsets V_1, V_2, H_1 and H_2. Note that $< V_1 >= K_{n-2}$ and $< V_2 >= \frac{n}{2}K_2$ in $\overline{\Gamma}$. Also, no element from V_1 is adjacent to any element to V_2 in $\overline{\Gamma}$ and vice-versa. This implies that $< S >$ is either a complete subgraph or a disconnected subgraph of $\overline{\Gamma}$ and hence $\overline{\Gamma}$ contains no induced cycle subgraph of odd order greater than 5.

Suppose n is odd. Let $V_1 = \{r, r^2, \ldots, r^{n-1}\}$ and $V_2 = \{s, sr, sr^2, \ldots, sr^{n-1}\}$. Note that $< V_1 > \cong K_{n-1}$ and $< V_2 >= \overline{K_n}$ in Γ. Also, no element from V_1 is adjacent to any element to V_2 in $\overline{\Gamma}$ and vice-versa. This implies that $< S >$ is either a complete subgraph or a disconnected subgraph of $\overline{\Gamma}$ and hence $\overline{\Gamma}$ contains no induced cycle subgraph of odd order greater than 5.

Thus neither Γ nor $\overline{\Gamma}$ contains an induced cycle subgraph of odd order greater than 5 and Γ is a perfect graph. □

In the following theorem, we obtain the clique covering number of Γ.

Theorem 2.14. *Let $n \geq 3$ be an integer. Then,*

$$\theta(\Gamma) = \begin{cases} n-2 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}.$$

Proof. **Case (i).** Let n be even.
As given in Theorem 2.4, consider the subsets V_1, V_2, H_1 and H_2. Since $< H_1 >=< H_2 >= K_{\frac{n}{2}}$ in Γ and every element in V_1 is adjacent to every element to V_2 in Γ, $< H_1 \cup \{x\} >=< H_2 \cup \{y\} >= K_{\frac{n}{2}+1}$ in Γ, where x and y are distinct elements in V_1. Then $< V_1 \setminus \{x, y\} >= \underset{n-4}{\cup} K_1$. Hence $\theta(\Gamma) = n - 2$.

Case (ii). Let n be odd.
Let $V_1 = \{r, r^2, \ldots, r^{n-1}\}$ and $V_2 = \{s, sr, sr^2, \ldots, sr^{n-1}\}$. Note that $< V_2 >= K_n$ in Γ and every element of $x \in V_1$ is adjacent to every element from V_2 in Γ. Since $< V_2 \cup \{x\} >= K_{n+1}$ in Γ where $x \in V_1$, we have $< V_1 \setminus \{x\} >= \underset{n-2}{\cup} K_1$ in Γ and hence $\theta(\Gamma) = n - 1$. □

In the following theorem, we discuss the genus nature of the graph Γ.

Theorem 2.15. *Let $n \geq 3$ be an integer. Then,*

$$g(\Gamma) \geq \begin{cases} g(K_{n-2,n}) & \text{if } n \text{ is even ;} \\ g(K_{n-1,n}) & \text{if } n \text{ is odd.} \end{cases}$$

Proof. **Case (i).** Let n be even.
As given in Theorem 2.4, consider the subsets V_1, V_2. Then Γ contains a subgraph $K_{n-2,n}$ by the vertex partitions V_1, V_2. By Theorem 1.1, $g(\Gamma) \geq g(K_{n-2,n})$.
Case (ii). Let n be odd.
Let $V_1 = \{r, r^2, \ldots, r^{n-1}\}$ and $V_2 = \{s, sr, sr^2, \ldots, sr^{n-1}\}$. Then Γ contains a subgraph $K_{n-1,n}$ by the vertex partitions V_1 and V_2. By Theorem 1.1, $g(\Gamma) \geq g(K_{n-1,n})$. □

In the following theorem, we discuss the toroidal nature of Γ .

Theorem 2.16. *Let $n \geq 3$ be an integer. Then Γ is non toroidal.*

Proof. **Case (i).** Let n be even.
As given in Theorem 2.4, consider the subsets V_1, V_2. If $G = D_8$, then by [[1],Proposition 2.3], Γ is planar. Suppose $n \geq 12$, then $|V_1| \geq 4$ and $|V_2| \geq 6$. Now Γ contains a subgraph $K_{4,6}$ and by Theorem 1.1 , Γ is non toroidal.
Case (ii). Let n be odd.
Let $V_1 = \{r, r^2, \ldots, r^{n-1}\}$ and $V_2 = \{s, sr, sr^2, \ldots, sr^{n-1}\}$. If $G = D_6$, then by [[1],Proposition 2.3] , Γ is planar. Suppose $n \geq 10$, then $|V_1| \geq 4$ and $|V_2| \geq 5$. Now Γ contains a subgraph $K_{4,5}$ and by Theorem 1.1 , Γ is non toroidal. □

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