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PROPERTIES OF GRAPHS FROM COMMUTATIVE RING

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Abstract

Let R be a commutative ring and $Z(R)^*$ be its set of all nonzero zero- divisors. The *zero-annihilator graph* of a commutative ring R is the simple undirected graph $\Gamma_Z(R)$ with vertices $Z(R) \setminus J(R)$, and two distinct vertices x and y are adjacent if and only if $ann(x) \cap ann(y) = \{0\}$. In this paper, we study some basic algebraic and graph theoretical properties of $\Gamma_Z(R)$.
Keywords: commutative ring, annihilator graph, empty graph, split graph.

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1 Introduction

In [9], Beck associated to a ring R its zero-divisor graph $G(R)$ whose vertices are the zero-divisors of R (including 0), and two distinct vertices x and y are adjacentif xy is zero. Later, in [5], Anderson and Livingston studied the subgraph $\Gamma(R)$ (of $G(R)$) whose vertices are the nonzero zero-divisors of R. In the recent years, several researchers have done interesting and enormous works on this field of study.For instance, see [1, 3, 7, 8, 10, 16]. The concept of co-annihilating ideal graph of a ring R, denoted by AR was introduced by Akbari et al. in [2]. As in [2], co-annihilating ideal graph of R is a graph whose vertex set is the set of all non- zero proper ideals of R and two distinct vertices I and J are adjacent whenever $Ann_R(I) \cap Ann_R(J) = \{0\}$. In [15], H. Mostafanasab have introduced and studiedthe zero-annihilator graph of R denoted by $\Gamma_Z(R)$. It is the graph whose vertex setis the set of all nonzero nonunit elements of R and two distinct vertices x and y are adjacent whenever $Ann_R(Rx+Ry) = Ann_R(x) \cap Ann_R(y) = \{0\}$. For basic definitions on rings, one may refer [6, 13, 14].

Let $G = (V, E)$ be a simple graph. Let $u, v \in V(G)$, define $d(x, y)$ to be the length of a shortest path from u to v in G. The diameter of G is $diam(G) = \sup\{d(x, y) : x, y \in V\}$. The girth of G, denoted by $gr(G)$ is the length of a shortest cycle in G. A complete bipartite graph with part sizes m and n is denoted by $K_{m,n}$. An undirected graph is an outerplanar graph if it can be drawn in the plane without crossings in sucha way that all of the vertices belong to the unbounded face of the drawing. There is a characterization for outerplanar graphs that says that a graph is outerplanar if and only if it does not contain a subdivision of K_4 or $K_{2,3}$. A graph is said to be planar ifit can be drawn in the plane so that its edges intersect only at their ends. For basic definitions on graphs, one may refer [12].

2 BASIC PROPERTIES OF $\Gamma_Z(R)$

In this section, we prove that $\Gamma_Z(R)$ is connected with diameter at most three. Further, we classify all commutative ring R for which $\Gamma_Z(R)$ is split. Note that if R is local ring, then $\Gamma_Z(R)$ is empty.

Theorem 2.1. *Let R be an Artinian ring which contains n maximal ideals. Then $\Gamma_Z(R)$ is connected with diam($\Gamma_Z(R)$)≤ 3.*

Proof. Let $u, v \in Z(R) \setminus J(R)$. If $u \in \mathfrak{m}_i, v \in \mathfrak{m}_j$ and $u, v \notin \mathfrak{m}_i \cap \mathfrak{m}_j$, then $u - v$ is a path. If $u, v \in \mathfrak{m}_i$, then there exists $w \in \mathfrak{m}_j$ such that $v - w - u$ is a path of length 2. If $u \in \mathfrak{m}_i \setminus \mathfrak{m}_j, v \in \mathfrak{m}_i \cap \mathfrak{m}_j$, then there exists $w \in \mathfrak{m}_k \setminus \mathfrak{m}_i \cup \mathfrak{m}_j$, such that $v - w - u$ is a path of length 2. If $u \in \cap \mathfrak{m}_i \setminus \mathfrak{m}_k, v \in \cap \mathfrak{m}_j \setminus \mathfrak{m}_i, i \neq k$, then there existx $c \in \mathfrak{m}_k$ and $d \in \mathfrak{m}_i$ such that $u - c - d - v$ is a path of length 3. Hence $diam(\Gamma_Z(R))\leq 3$. □

Theorem 2.2. *Let R be an Artinian ring which contains n maximal ideals. Then $\Gamma_Z(R)$ is a complete bipartite graph if and only if $R \cong R_1 \times R_2$, where $(R_i, \mathfrak{m}_i)$ is a local ring.*

Proof. Suppose $\Gamma_Z(R)$ is complete bipartite graph. Since R is Artinian, $R \cong R_1 \times \cdots \times R_n$ where $(R_i, \mathfrak{m}_i)$ is a local ring for $1 \leq i \leq n$. If $n \geq 3$, then $(1, 1, 0, 1, \ldots, 1) - (1, 0, 1, 1, \ldots, 1) - (0, 1, 1, \ldots, 1) - (1, 1, 0, 1, \ldots, 1)$ is a cycle of length 3, which is a contradiction. Hence $n = 2$..

Conversely, If $R \cong R_1 \times R_2$, where $(R_i, \mathfrak{m}_i)$ is local. Note that $V(\Gamma_Z(R))$= $\{(x, b), (a, y) : x \in R_1^\times, y \in R_2^\times, a \in \mathfrak{m}_1, b \in \mathfrak{m}_2\}$. Also two vertices (x, b) and (a, y) are adjacent. On the other hand, every two vertices (d_1, b_1), (d_2, b_2) cannot be adjacent. Similarly for (c_1, v_1) and (c_2, v_2). □

Theorem 2.3. *Let R be an Artinian ring which contains n maximal ideals. Then the following are equivalent:*

(a) $\Gamma_Z(R)$ *is a star.*

(b) $\Gamma_Z(R)$ *is a tree.*

(c) $R \cong \mathbb{Z}_2 \times F$ *where F is a field.*

Proof. $(a) \Rightarrow (b)$ follows from the definition of tree.
$(b) \Rightarrow (c)$ Suppose $\Gamma_Z(R)$ is a tree. Since R is finite, $R = R_1 \times \cdots \times R_n$ where each $(R_i, \mathfrak{m}_i)$ is a local ring. If $n \geq 3$, then $(1,1,0,1,\ldots,1)-(1,0,1,\ldots,1)-(0,1,1,\ldots,1)-(1,1,0,1,\ldots,1)$ is a cycle in $\Gamma_Z(R)$, a contradiction. Thus $n = 2$ and $R = R_1 \times R_2$. If $|\mathfrak{m}_1^*| \geq 1$, then $(u_1,0)-(0,1)-(u_2,0)-(a,1)-(u_1,0)$, where $u_1, u_2 \in R_1^{\times}, a \in \mathfrak{m}_1^*$ is a cycle in $\Gamma_Z(R)$, a contradiction. Hence R_1 and R_2 are fields. IF $|R_i| \geq 3$ for all i, then $(u_1,0)-(0,v_1)-(u_2,0)-(0,v_2)-(u_1,0)$, where $u_1, u_2 \in R_1^*$ and $v_1, v_2 \in R_2^*$ is a cycle, a contradiction. Hence $|R_i| = 2$ for some i and therefore $R \cong \mathbb{Z}_2 \times F$, where F is a field.
$(c) \Rightarrow (a)$ If $R \cong \mathbb{Z}_2 \times F$, where F is a field, then $\Gamma_Z(R) \cong K_{1,|F^*|}$. □

Theorem 2.4. *Let R be an Artinian ring which contains n maximal ideals. Then*

(a) $gr(\Gamma_Z(R)) = 3$ *if and only if* $|Max(R)| \geq 3$.

(b) $gr(\Gamma_Z(R)) = 4$ *if and only if* $|Max(R)| = 2$ *and* $R \ncong \mathbb{Z}_2 \times F$ *where F is a finite field.*

(c) $gr(\Gamma_Z(R)) = \infty$ *if and only if* $R \cong \mathbb{Z}_2 \times F$ *where F is a finite field.*

Proof. Proof follows from above theorems. □

A split graph is a simple graph in which the vertices can be partitioned into a clique and an independent set. There is a characterization for split graphs that says that a graph is a split graph if and only if it contains no induced subgraph isomorphic to $2K_2, C_4, C_5$.

Theorem 2.5. *Let R be an Artinian ring which contains n maximal ideals. Then $\Gamma_Z(R)$ is a split graph if and only if R is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{F}$, where F is a field.*

Proof. Since R is finite, $R = R_1 \times R_2 \times \cdots \times R_n$ where each R_i is local for $1 \leq i \leq n$. Clearly $n \geq 2$. Suppose $\Gamma_Z(R)$ is split graph.

Suppose $n \geq 4$. Then there exist vertices $(1,0,1,1,1,\ldots,1)$, $(0,1,1,0,1\ldots,1)$, $(1,0,0,1,\ldots,1), (0,1,1,1,1,\ldots,1) \in Z(R) \setminus J(R)$ which makes C_4 in $\Gamma_Z(R)$, which is a contradiction. Hence $n \leq 3$.

Suppose $n = 3$ and $|R_i| \geq 3$ for some i. Without loss of generality, assume that $|R_3| \geq 3$. Then there exist vertices $x_1 = (1,1,0)$, $x_2 = (0,1,1)$, $x_3 = (1,0,0)$, $x_4 = (0,1,a) \in Z(R) \setminus J(R)$, where $a \in R_3^*$ make C_4 as a subgraph in $\Gamma_Z(R)$, which is a contradiction. Thus $|R_i| = 2$ for all i and so $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Suppose $n = 2$ and $\mathfrak{m}_i \neq \{0\}$ for some i. Assume that $\mathfrak{m}_1 \neq \{0\}$. Then $|R_1| \geq 4$. Consider $x_1 = (u_1, 0)$, $x_2 = (0,1)$, $x_3 = (u_2, 0)$, $x_4 = (a, 1) \in Z(R) \setminus J(R)$, where $a \in \mathfrak{m}_1^*$, $u_1, u_2 \in R_1^\times$. Then $\{x_1, \ldots, x_4\}$ induced a subgraph which is isomorphic to C_4 in $\Gamma_Z(R)$, which is a contradiction. Thus R_i is field for all i. Since $\Gamma_Z(R) \cong K_{|R_1^*|,|R_2^*|}$ and so $\Gamma_Z(R)$ is split graph, $R \cong \mathbb{Z}_2 \times F$ where F is a field. □

Theorem 2.6. *Let R be an Artinian ring which contains n maximal ideals. Then $\Gamma_Z(R) \cong \Gamma(R)$ if and only if $R \cong \mathbb{Z}_2^n$.*

Proof. Suppose $\Gamma_Z(R) \cong \Gamma(R)$. If $\mathfrak{m}_i \neq \{0\}$ for some i, then $|Z(R) \setminus J(R)| < |Z(R)^*|$. Hence $\mathfrak{m}_i = \{0\}$ for all i and so R_i is field for $1 \leq i \leq n$. Suppose $|R_i| \geq 3$ for some i. It is clear that $|E(\Gamma(R)| < |E(\Gamma_Z(R))|$. Hence $|R_i| = 2$ for all i and $R \cong \mathbb{Z}_2^n$.

Conversely, suppose $R \cong \mathbb{Z}_2^n$. Let $x = (x_1, x_2 \ldots, x_n) \in Z(R)^*$. Define $\Psi : Z(R)^* \to Z(R)^*$ by

$$\Psi(x) = \begin{cases} 1 & \text{if} \quad x_i = 0 \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$$

Clearly Ψ is bijective. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in Z(R)^*$. If x and y are adjacent in $\Gamma_Z(R)$, then $ann_R(x) \cap ann_R(y) = \{0\}$ and so $ann_{R_i}(x_i) \cap ann_{R_i}(y_i) = \{0\}$. Clearly $x_i y_i = 0$ for all i and so $xy = 0$. Similarly Ψ preserves non adjacency also. Thus $\Gamma_Z(R) \cong \Gamma(R)$. □

3 Planar property of $\Gamma_Z(R)$

In this section, we determined the class of rings for which $\Gamma_Z(R)$ is planar.

Theorem 3.1. *Let R be an Artinian ring which contains n maximal ideals. Then $\Gamma_Z(R)$ is outerplanar if and only if R is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times F$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, or $\frac{\mathbb{Z}_2[x]}{<x^2>} \times \mathbb{Z}_2$, where F is a field.*

Proof. Since R is Artinian, $R = R_1 \times R_2 \times \cdots \times R_n$, where each R_i is local for $1 \leq i \leq n$. Clearly $n \geq 2$. Assume that $\Gamma_Z(R)$ is outerplanar.

Suppose $n \geq 4$. Then there exist vertices $(1,1,1,0,1,\ldots,1)$, $(1,1,0,1\ldots,1)$, $(1,0,1,1,\ldots,1), (0,1,1,\ldots,1) \in Z(R)\backslash J(R)$ which makes K_4 as a subgraph in $\Gamma_Z(R)$, a contradiction. Thus $n \leq 3$.

Suppose $n = 3$ and $|R_i| \geq 3$ for some i. Without loss of generality, assume that $|R_3| \geq 3$. Let $x_1 = (1,1,0)$, $x_2 = (1,0,1)$, $x_3 = (0,1,1)$, $x_4 = (1,0,a)$, $x_5 = (0,1,0)$, where $a \in R_3^*$ in $\Gamma_Z(R)$. Then $\{x_1,\ldots,x_5\}$ induces a subgraph which contains a subdivision of K_4 in $\Gamma_Z(R)$, a contradiction. Thus $|R_i| = 2$ for all i and so $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

If $n = 2$ and $\mathfrak{m}_i \neq \{0\}$ for all i, then $|R_i| \geq 4$ for $i = 1,2$. Consider $x_1 = (u_1,0)$, $x_2 = (u_2,0)$, $x_3 = (0,v_1)$, $x_4 = (0,v_2)$, $x_5 = (a,v_1) \in Z(R) \setminus J(R)$, where $a \in \mathfrak{m}_1^*$, $u_1,u_2 \in R_1^\times$ and $v_1,v_2 \in R_2^\times$. Then $\{x_1,\ldots,x_5\}$ induces a subgraph which is isomorphic to $K_{2,3}$ in $\Gamma_Z(R)$, which is a contradiction. Thus at least one of the R_i is local with $\mathfrak{m}_i = \{0\}$. Suppose that $\mathfrak{m}_2 = \{0\}$.

Suppose $|\mathfrak{m}_1^*| \geq 2$. Clearly $|R_1^\times| \geq 4$. Let $x_1 = (u_1,0)$, $x_2 = (u_2,0)$, $x_3 = (u_3,0)$, $x_4 = (u_4,0)$, $x_5 = (0,1)$, $x_6 = (a,1) \in Z(R) \setminus J(R)$, where $u_i \in R_1^\times, a \in \mathfrak{m}_1^*$. Then $\{x_1,\ldots,x_6\}$ make $K_{4,2}$ as a subgraph in $\Gamma_Z(R)$, which is a contradiction. Therefore $|\mathfrak{m}_1^*| = 1$ or 0 and so R_1 is either field or $R_1 \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{<x^2>}$.

Suppose $R = R_1 \times R_2$ where R_i's are field. Since $\Gamma_Z(R) \cong K_{|R_1^*|,|R_2^*|}$, $R \cong \mathbb{Z}_2 \times F$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$ where F is a field.

Suppose $R = R_1 \times R_2$ where R_i's are field. Since $\Gamma_Z(R) \cong K_{|R_1^*|,|R_2^*|}$, $R \cong \mathbb{Z}_2 \times F$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$ where F is a field.

Suppose $R = R_1 \times R_2$, where $R_1 \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{<x^2>}$ and R_2 is a field. If $|R_2| \geq 3$, then there exist vertices $x_1 = (u_1, 0)$, $x_2 = (u_2, 0)$, $x_3 = (0, 1)$, $x_4 = (0, a)$, $x_5 = (z, 1)$, where $u_1, u_2 \in R_1^\times$, $z \in \mathfrak{m}_1^*$ and $1 \neq a \in R_2^*$. Then $\{x_1, \ldots, x_5\}$ make a subgraph isomorphic to $K_{2,3}$ in $\Gamma_Z(R)$, which is a contradiction. Thus $|R_2| = 2$ and so $R \cong \mathbb{Z}_4 \times \mathbb{Z}_2$, or $\frac{\mathbb{Z}_2[x]}{<x^2>} \times \mathbb{Z}_2$. □

Theorem 3.2. *Let R be an Artinian reduced ring which contains $n \geq 2$ maximal ideals. Then $\Gamma_Z(R)$ is planar if and only if R is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times F$, $\mathbb{Z}_3 \times F$, where F is a field.*

Proof. Since R is reduced, $R = F_1 \times \cdots \times F_n$, where each F_i is field and $n \geq 2$. Suppose that $\Gamma_Z(R)$ is planar. Suppose $n \geq 4$. Consider $S = \{x_1, \ldots, x_6\}$, where $x_1 = (1, 1, 1, 0, 1, \ldots, 1)$, $x_2 = (1, 1, 0, 1, \ldots, 1)$, $x_3 = (1, 0, 1, 1, \ldots, 1)$, $x_4 = (0, 1, 1, 1, \ldots, 1)$, $x_5 = (1, 0, 1, 0, 1, \ldots, 1)$, $x_6 = (0, 1, 0, 1, 1, \ldots, 1) \in Z(R) \setminus J(R)$. Then the subgraph induced by S contains a contraction of K_5, a contradiction. Thus $n \leq 3$.

Suppose $n = 3$ and $|F_i| \geq 3$ for some i. Without loss of generality assume that $|F_3| \geq 3$. Let $x_1 = (1, 1, 0)$, $x_2 = (1, 0, 1)$, $x_3 = (1, 0, a)$, $x_4 = (0, 1, 1)$, $x_5 = (0, 1, a)$, $x_6 = (1, 0, 0)$, $x_7 = (0, 1, 0)$ where $1 \neq a \in F_3^*$. Then the subgraph induced by $\{x_1, \ldots, x_7\}$ is isomorphic to a subdivision of K_5, a contradiction. Therefore $|F_i| = 2$ for all i and hence $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Suppose $n = 2$. Then $R = F_1 \times F_2$. Clearly $\Gamma_Z(R) \cong K_{|F_1^*|,|F_2^*|}$. Therefore $|F_i| \leq 3$ at least one i. Hence $R \cong \mathbb{Z}_2 \times F$ or $\mathbb{Z}_3 \times F$, where F is a field. □

Theorem 3.3. *Let R be an Artinian non-reduced ring which contains $n \geq 2$ maximal ideals. Then $\Gamma_Z(R)$ is nonplanar.*

Proof. Since R is Artinian ring, $R = R_1 \times \cdots \times R_n$, where each R_i is local with $\mathfrak{m}_i \neq \{0\}$ and $n \geq 2$. Since $\mathfrak{m}_i \neq \{0\}$, $|R_i| \geq 4$ for all $1 \leq i \leq n$. Consider $S = \{x_1, \ldots, x_4, y_1, \ldots, y_4\}$, where $x_1 = (u_1, 0)$, $x_2 = (u_2, 0)$, $x_3 = (u_1, b)$, $x_4 = (u_2, b)$, $y_1 = (0, v_1)$, $y_2 = (0, v_2)$, $y_3 = (a, v_1)$, $y_4 = (a, v_2)$, $u_1, u_2 \in R_1^\times, v_1, v_2 \in R_2^\times, a \in \mathfrak{m}_1^*$ and $b \in \mathfrak{m}_2^*$. Then the subgraph induced by S is isomorphic to $K_{4,4}$, a contradiction, which completes the proof. □

Theorem 3.4. *Let $R = R_1 \times \cdots \times R_m \times F_1 \times \cdots \times F_n$, where R_i is a local ring with $\mathfrak{m}_i \neq \{0\}$ and F_j is a field. Then $\Gamma_Z(R)$ is planar if and only if R is isomorphic to $\mathbb{Z}_4 \times F$ or $\frac{\mathbb{Z}_2[x]}{<x^2>} \times F$, where F is a field.*

Proof. Assume that $\Gamma_Z(R)$ is planar. Then by Theorem 3.3, $m = 1$. Suppose $n \geq 2$. Consider $S = \{x_1, \ldots, x_7\}$, where $x_1 = (u_1, 1, 0)$, $x_2 = (u_2, 1, 0)$, $x_3 = (0, 1, 1)$, $x_4 = (u_1, 0, 1)$, $x_5 = (u_2, 0, 1)$, $x_6 = (0, 0, 1)$, $x_7 = (1, 0, 0)$, where $u_1, u_2 \in R_1^\times$. Then the subgraph induced by S contains a subdivision of K_5, which is a contradiction. Hence $n = 1$ and $R = R_1 \times F_1$.

Suppose $|\mathfrak{m}_1^*| \geq 2$, then $|R_1^\times| \geq 4$. Let $x_1 = (u_1, 0)$, $x_2 = (u_2, 0)$, $x_3 = (u_3, 0)$, $x_4 = (u_4, 0)$, $y_1 = (0, 1)$, $y_2 = (a, 1)$, $y_3 = (b, 1)$, where $u_i \in R_1^\times, 1 \leq i \leq 4$, $a, b \in \mathfrak{m}_1^*$. It is clear that each x_i is adjacent to y_j for $1 \leq i \leq 4$, $1 \leq j \leq 3$. Therefore $\Gamma_Z(R)$ contains $K_{4,3}$ as a subgraph, a contradiction. Hence $|\mathfrak{m}_1^*| = 1$ and $R_1 \cong \mathbb{Z}_4$ or $\frac{Z_2[x]}{<x^2>}$.

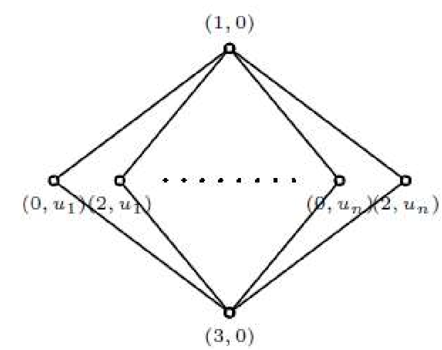

Figure 1: $\Gamma_Z(R)(\mathbb{Z}_4 \times F) \cong \Gamma_Z(R)(\frac{\mathbb{Z}_2[x]}{<x^2>} \times F)$

Converse follows from Figure 1. □

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