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THE FORCING DETOUR EDGE SEMI-TOLL NUMBER OF A GRAPH

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Abstract

Let $G = (V, E)$ be an undirected connected graph. Let S be a minimum detour edge semi-toll set (EST_{dn}-set) of G. A subset $M \subseteq S$ is said to be a forcing subset of S if S is the unique EST_{dn}-set containing M. The forcing EST_{dn} number $f_{EST_{dn}}(S)$ of S in G is the minimum cardinality of a forcing subset for S. The forcing detour edge semi-toll number $f_{EST_{dn}}(G)$ of G is the minimum cardinality of $f_{EST_{dn}}(S)$, where the minimum is taken over all EST_{dn}-sets S of G . It proved that $0 \leq f_{EST_{dn}}(G) \leq EST_{dn}(G)$ .Some general properties satisfied by this concept are studied. The forcing detour edge semi- toll number of some standard graphs are determined. Necessary and sufficient conditions for $f_{EST_{dn}}(G)$ to be 0 or 1 are characterized. It is shown that every pair of integers a and b with $0 \leq a \leq b$, there exists a connected graph G such that $f_{EST_{dn}}(G) = a$ and $EST_{dn}(G) = b$. Also it is shown that for every pair of integers a and b with $0 \leq a \leq b$, there exists a connected graph G such that $f_{EST_{dn}}(G)$ = a and $f_{dn}(G) = b$.

Keywords: detour number, forcing detour number, tolled walk, detour edge semi-toll number, forcing detour edge semi toll number.

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1. Introduction

By a graph $G = (V, E)$,we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by n and m respectively. For basic graph theoretic terminology, we refer to [2]. Two vertices u and v are said to be adjacent if uv is an edge of G. Two edges of G are said to be adjacent if they have a common vertex. A walk is defined as a finite length of alternating sequence of vertices and edges. The total number of edges covered in a walk is called as length of the walk. It is a trail in which neither vertices nor edges are repeated i.e. if we traverse a graph such that we do not repeat a vertex and nor we repeat an edge. Any connected graph is called as an Euler Graph if and only if all its vertices are of even degree. If there exists a walk in the connected graph that starts and ends at the same vertex and visits every edge of the graph exactly once with or without repeating the vertices, then such a walk is called as an Euler circuit. If a connected graph contains an Euler trail but does not contain an Euler circuit, then such a graph is called as a semi-Euler graph. The distance $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest u-v path in G. An $u{-}v$ path of length $d(u,v)$ is called an $u{-}v$ geodesic. The detour distance $D(u,v)$ between two vertices u and v in a connected graph G from u to v is defined as the length of a longest u-v path in G. An u-v path of length $D(u,v)$ is called an u-v detour. A vertex x is said to lie on an u-v detour P if x is a vertex of P including the vertices u and v. A detour set of G is a set $S \subseteq V\ (G)$ such that every vertex of G is contained in a detour joining some pair of vertices in S. The closed detour $I_D[u,v]$ consists of all the vertices lying on some u-v detour of G including the vertices u and v. The detour number $dn(G)$ of G is the minimum order of a detour set and any detour set of order $dn(G)$ is called minimum detour set of G or a dn-set of$G$. These concept were studied in [3-7].

A tolled walk T between u and v in G in a sequence at vertices of the form

$T: u, w_1, w_2, \dots, v$ where $k \geq 1$ which enjoys the following three conditions.

- $w_i w_{i+1} \in E(G), \forall\ i$
- $uw_i \in E(G)$ if and only if $i = 1$.
- $vw_i \in E(G)$ if and only if $i = k$.

$T[u,v]$ = set of vertices lying in the $u - v$ tolled walk including u and v.

A longest u-v tolled walk is called a *u-v detour tolled walk* $T_D[u,v]$. Set of all vertices lying in u-v detour tolled walk including u and v. For $S \subseteq V(G)$, the detour tolled closure of G is $T_D[S] = \bigcup_{u,v \in S} T_D[u,v]$. A set $S \subseteq V(G)$ is called a detour tolled set if $T_D[S] = V[G]$. The minimum cardinality of a detour tolled set is called the detour tolled number of G and is denoted by $t_{dn}(G)$.

An u-v walk P is called an edge semi tolled walk if no edge of $E[P]$ is repeated. A longest *u-v edge semi tolled walk* is called a u-v detour edge semi tolled walk. For two vertices $u, v \in V, EST_D[u,v] =$ set of all vertices lying in a u-v detour edge semi tolled walk. For $m \subseteq V, EST_D[M] = \bigcup_{u,v \in M} EST_D[u,v]$. A set $M \subseteq V$ is called a detour edge semi-toll set if $EST_D[M] = V[G]$. The minimum cardinality of a detour edge semi-toll set is called the detour edge semi-toll number of $G$and is denoted by $EST_{dn}(G)$. These concepts were studied in [1,11].The forcing concepts in graph were studied in [8-10] . In this article, we introduced a new concept called the forcing detour edge semi-toll number of a graph and some of its properties. These concepts were applied in communication networks.

Theorem 1.1 [1,4] Each end vertex of a connected graph G belongs to every detour (detour semi-toll) set of G.

Theorem 1.2 [4] Let G be a connected graph and W be the set of all detour vertices of G. Then $f_{dn}(G) \leq dn(G) - |W|$.

Theorem 1.3 [4] For the star graph $G = K_{1,n-1} (n \geq 4)$, $dn(G) = n - 1$ and $f_{dn}(G) = 0$.

2.The forcing detour edge semi-toll number of a graph

Definition 2.1. Let S be an EST_{dn}-set of G. A subset $M \subseteq S$ is said to be a forcing subset of S if S is the unique EST_{dn}-set containing M. The forcing EST_{dn} number $f_{EST_{dn}}(S)$ of S in G is the minimum cardinality of a forcing subset for S. The forcing detour edge semi- toll number $f_{EST_{dn}}(G)$ of G is the minimum cardinality of $f_{EST_{dn}}(S)$, where the minimum is taken over all EST_{dn}-sets S of G.

Example 2.2. For the graph G given in Figure 2.1, S_1= $\{v_1, v_3, v_{12}\}$, S_2= $\{v_1, v_4, v_{12}\}$ are the only two minimum EST_{dn}-sets of G such that $f_{EST_{dn}}(S_1) = f_{EST_{dn}}(S_2) = 1$ so that $f_{EST_{dn}}(G)$= 1.

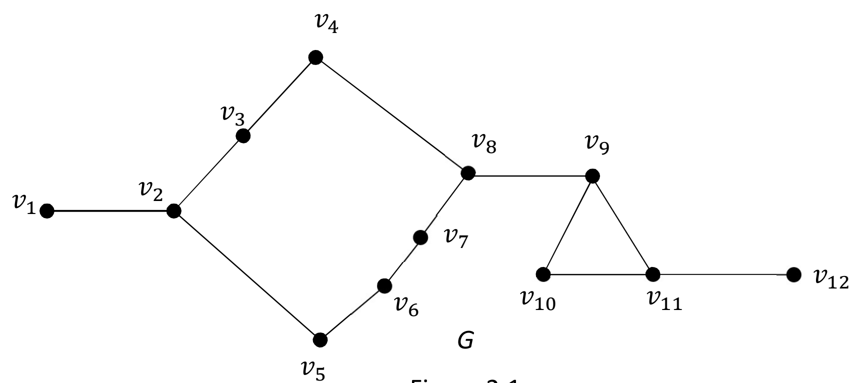


Figure 2.1

A graph G with $f_{EST_{dn}}(G) = 1$

Definition: 2.3. A vertex v of a graph G is said to be *edge semi-toll vertex* of$G$ if v belongs to every EST_{dn}-set of$G$.

Observation 2.4. For every connected graph$G$, $0 \leq f_{EST_{dn}}(G) \leq EST_{dn}(G)$.

Observation 2.5. Let G be a connected graph and W be the set of all edge semi-toll vertices of G. Then$f_{EST_{dn}}(G) \leq EST_{dn}(G) - |W|$.

Observation 2.6. Let G be a connected graph. Then

(a) $f_{EST_{dn}}(G) = 0$ if and only if G has a unique EST_{dn}-set.

(b) $f_{EST_{dn}}(G) = 1$ if and only if G has at least two EST_{dn}-sets, one of which is a unique EST_{dn}-set containing one of its elements, and

(c)$f_{EST_{dn}}(G) = EST_{dn}(G)$ if and only if no EST_{dn}-set of G is the unique EST_{dn}-set containing any of its proper subsets.

Observation 2.7. For the star graph G= $K_{1,n-1}$, $f_{EST_{dn}}(G) = 0$.

Observation 2.8. For the graph G= $K_{1,n-1} + e$, $f_{EST_{dn}}(G) = 1$.

Proof. Let $V(K_{1,n-1}) = \{x, v_1, v_2, \dots, v_{n-1}\}$ where x is the cut vertex of $K_{1,n-1}$ and $e = v_1v_2$. Then $S_1 = \{v_1, v_3, v_4, \dots, v_{n-1}\}$ and $S_2 = \{v_2, v_3, \dots, v_{n-1}\}$ are the only two EST_{dn}-sets of G so that $f_{EST_{dn}}(G) = 1$. ■

Theorem 2.9. For the complete graph $G= K_n(n \geq 3), f_{EST_{dn}}(G) = 2$.

Proof. Let $S = \{u, v\}$ be set of two adjacent vertices of G. Then S is a EST_{dn}-set of G so that $EST_{dn}(G) = 2$. Since $n \geq 3, EST_{dn}$-set of G is not unique and so $f_{EST_{dn}}(G) \geq 1$. Since $n \geq 3, u$ and v lie on two different EST_{dn}-sets of G and so $f_{EST_{dn}}(G) = 2$. Since this is true for all EST_{dn}-sets S of G, it follows that $f_{EST_{dn}}(G) = 2$. ■

Theorem 2.10. For the complete bipartite graph $G= K_{m,n}, m \geq 2, n \geq 2, f_{EST_{dn}}(G) = 2$.

Proof. Let $S = \{u, v\}$ be set of two adjacent vertices of G. Then S is a EST_{dn}-set of G so that $EST_{dn}(G) = 2$. Since $n \geq 3, EST_{dn}$-set of G is not unique and so $f_{EST_{dn}}(G) \geq 1$. Since $m \geq 2, n \geq 2, u$ and v lie on two different EST_{dn}-sets of G and so $f_{EST_{dn}}(G) = 2$. Since this is true for all EST_{dn}-sets S of G, it follows that $f_{EST_{dn}}(G) = 2$. ■

Theorem 2.11. For the cycle graph $G = C_n, n \geq 3, f_{EST_{dn}}(G) = 2$.

Proof. Let $S = \{u, v\}$ be set of two adjacent vertices of G. Then S is a EST_{dn}-set of G so that $EST_{dn}(G) = 2$. Since $n \geq 4, EST_{dn}$-set of G is not unique and so $f_{EST_{dn}}(G) \geq 1$.Since $n \geq 3, u$ and v lie on two different EST_{dn}-sets of G and so $f_{EST_{dn}}(G) = 2$. Since this is true for all EST_{dn}-sets S of G, it follows that $f_{EST_{dn}}(G) = 2$. ■

Theorem 2.12. For the wheel graph $G = K_1 + C_{n-1}\ (n \geq 4),\ f_{EST_{dn}}(G) = 2$.

Proof. Let $S = \{u, v\}$ be set of two adjacent vertices of G. Then S is a EST_{dn}-set of G so that $EST_{dn}(G) = 2$. Since $n \geq 4, EST_{dn}$-set of G is not unique and so $f_{EST_{dn}}(G) \geq 1$. Since $n \geq 4, u$and v lie on two different EST_{dn}-setsof G and so $f_{EST_{dn}}(G) = 2$. Since this is true for all EST_{dn}-sets S of G, it follows that $f_{EST_{dn}}(G) = 2$. ■

Theorem 2.13. For the non-trivial graph T tree, $f_{EST_{dn}}(G) = 0$.

Proof. Since the set of all end vertices of T is a EST_{dn}-sets of G so that $EST_{dn}(G) = 2$.

The result follows that $f_{EST_{dn}}(G) = 0$. ■

Theorem 2.14. For every pair of integers a and b with $0 \leq a \leq b$, there exists a connected graph G such that $f_{EST_{dn}}(G) = a$ and $EST_{dn}(G) = b$.

Proof. For $1 \leq i \leq a$, let $H_i: u_i, v_i, w_i$ be a copy of the complete graph K_3. Let J_a be the graph obtained from $H_i(1 \leq i \leq a)$ by adding new vertex x and introducing the edge $xu_i(1 \leq i \leq a)$. Let G be the graph obtained from J_a by adding new vertices $z_1, z_2, \dots, z_{b-a}$ and introducing the edge $xz_i(1 \leq i \leq b - a)$. The graph G is shown in Figure 2.2.

First we prove that $EST_{dn}(G) = b$. Let $Q_i = \{v_i, w_i\}, (1 \leq i \leq a)$ and $Z = \{z_1, z_2, \ldots, z_{b-a}\}$. It is easily observed that every EST_{dn}-set of G contains each $z_i (1 \leq i \leq b-a)$ and atleast one vertex from each and so $EST_{dn}(G) \geq b - a + a = b$. Let $S = Z \cup \{v_1, v_2, \ldots, v_a\}$. Then S is a detour edge semi toll set of G so that $EST_{dn}(G) = b$.
Next we prove that $f_{EST_{dn}}(G) = a$. By Observation 2.5, $f_{EST_{dn}}(G) \leq EST_{dn}(G) - |Z| = b - (b-a) = a$. Since Z is a subset of every EST_{dn}-set of G and every EST_{dn}-set of G contains exactly one vertex from $Q_i (1 \leq i \leq a)$ every EST_{dn}-set S is of the form $S = Z \cup \{c_1, c_2, \ldots, c_a\}$ where $c_i \in Q_i (1 \leq i \leq a)$. We show that $f_{EST_{dn}}(G) = a$. On the contrary suppose that $f_{SE_{dn}}(G) < a$. Then there exists a forcing subset T of S such that $|T| < a$. This show that $T \cap S = \phi$, which is a contradiction. Therefore $f_{EST_{dn}}(G) = a$.

■

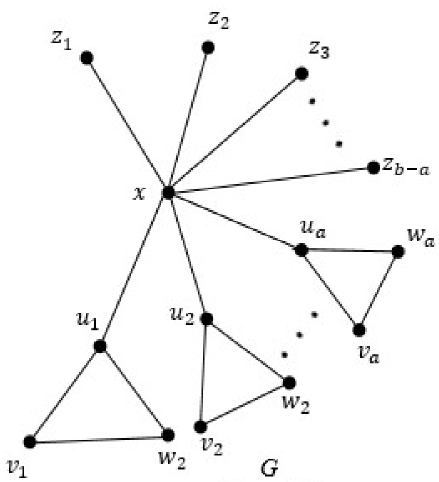


Figure 2.2
A graph G with $f_{EST_{dn}}(G) = a$

Theorem 2.15. For every pair of integers a and b with $0 \leq a \leq b$ and $b \geq 2$, there exists a connected graph G such that $f_{EST_{dn}}(G) = 0, f_{dn}(G) = a$ and $dn(G) = b$.

Proof. For $a = 0$ and $b \geq 2$, let $G = K_{1,b}$. Then by Observation 2.7 $f_{EST_{dn}}(G) = 0$,

$f_{dn}(G) = 0$ and $dn(G) = b$. For $a = 1$ and $b \geq 2$, Consider the Figure 2.3 It is easily verified that, $f_{EST_{dn}}(G) = 0, f_{dn}(G) = 1$ and $dn(G) = b$. So, let $2 \leq a < b$. Let $P_i: u_i, v_i (1 \leq i \leq a)$ be a copy of path on two vertices. Let $H_a$be the graph obtained from $P_i (1 \leq i \leq a)$ by adding new vertex y and introducing the edges x_i and $y_i (1 \leq i \leq a)$. Let $H(a, b-a)$ be the graph obtained from H_a by adding new vertices $z_1, z_2, \ldots, z_{b-a}$ and introducing the edge $yz_i (1 \leq i \leq b-a)$. The graph $G = H(a, b-a)$ is shown in Figure 2.4.

First we prove that $f_{EST_{dn}}(G) = 0$. Let $Z = \{z_1, z_2, \ldots, z_{b-a}\}$ be the set of end vertices of G. By Theorem 1.1, Z is a subset of every detour edge semi toll set of G. Now Z is the unique detour edge semi toll set of G so that $f_{EST_{dn}}(G) = 0$.

Next we prove that $dn(G) = b$. Let $H_i: \{x_i, y_i\}(1 \le i \le b - a)$ and $Z = \{z_1, z_2, \dots, z_{b-a}\}$. It is easily observed that every dn-set of G contains each $z_i(1 \le i \le b - a)$ and at least one vertex from each $H_i(1 \le i \le a)$ and so $dn(G) \ge b - a + a = b$. Let $S = Z \cup \{y_1, y_2, \dots, y_a\}$. Then S is a detour set of G so that $dn(G) = b$.

Next we prove that $f_{dn}(G) = a$. By Theorem 1.2, $f_{dn}(G) \le dn(G) - |Z| = b - (b - a) = a$. Since Z is a subset of every dn-set of G and every dn-set of G contains exactly one vertex from $H_i(1 \le i \le a)$ every dn-set S is of the form $S = Z \cup \{c_1, c_2, \dots, c_a\}$ where $c_i \in H_i(1 \le i \le a)$. We show that $f_{dn}(G) = a$. On the contrary suppose that $f_{dn}(G) < a$. Then there exists a forcing subset T of S such that $|T| < a$. This show that $T \cap S = \phi$, which is a contradiction. Therefore$f_{dn}(G) = a$. ∎

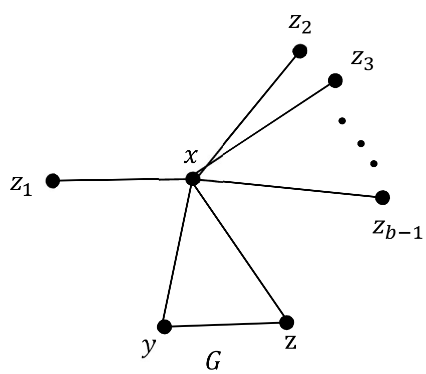


Figure 2.3

A graph G with $f_{EST_{dn}}(G) = 0$, $f_{dn}(G) = 1$ and $dn(G) = b$

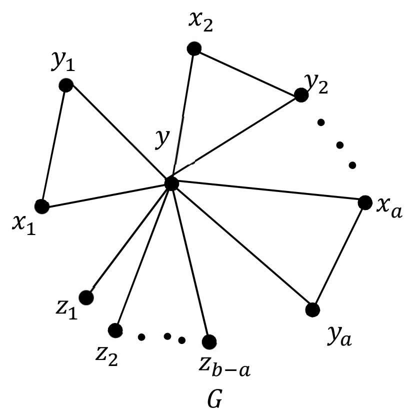


Figure 2.4

A graph G with $f_{EST_{dn}}(G) = 0$,

$f_{dn}(G) = a$ and $dn(G) = b$

Theorem 2.16. For every pair of positive integers a and b with $0 \le a \le b$, there exists a connected graph G such that $f_{EST_{dn}}(G) = a$ and $f_{dn}(G) = b$.

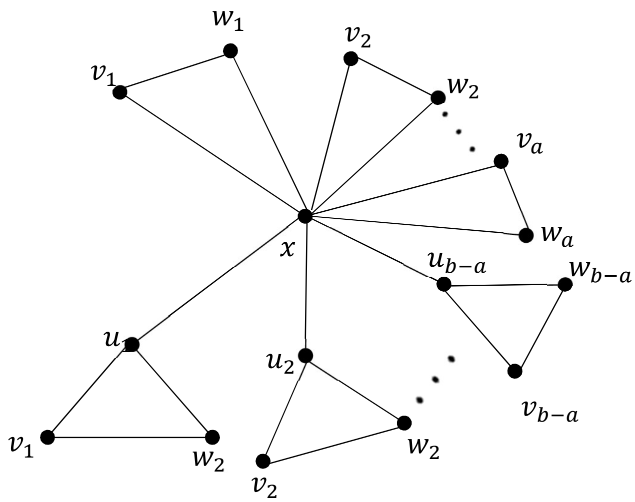
Proof. Let G be the graph obtained from $J_a$and H_{b-a} by identifying x of J_a and y of H_{b-a}.The graph G is shown in Figure 2.5.

First we prove that $f_{EST_{dn}}(G) = a$. Let $Q_i = \{v_i, w_i\}(1 \le i \le a)$. Then every EST_{dn}-set of G contains at least one vertex from each $Q_i(1 \le i \le b)$ and so $EST_{dn}(G) \ge a$. Let $S = \{v_1, v_2, \dots, v_a\}$. Then S is a EST_{dn}-set of G and so $EST_{dn}(G) = a$. Since no proper subset of S is a forcing subset of S, $f_{ES\ \ dn}(G) = a$. Since this is true for all S, $f_{EST_{dn}}(G) = a$.

Next we prove that $f_{dn}(G) = b$. Let $H_i = \{x_i, y_i\}(1 \le i \le b)$. Now every dn-set of G contains exactly one vertex from each $Q_i(1 \le i \le a)$ and exactly one vertex from each $H_i(1 \le i \le b - a)$ and so $dn(G) \ge b - a + a = b$. Let $M = S \cup \{x_1, x_2, \dots,$

$x_{b-a}\}$. Then M is a dn-set of G so that $dn(G) = b$. Since no proper subset of M is a forcing subset of M, $f_{dn}(M) = b$. Since this is true for all dn-set M of G, $f_{dn}(G) = b$.

∎



G

Figure 2.5

A graph G with $f_{EST_{dn}}(G) = a$ and

$f_{dn}(G) = b$

3. Conclusion

In this article, we define forcing detour edge semi-toll number of a graph and determine forcing detour edge semi-toll number of a graph for some standard graphs. Finally we present some realisation results.In subsequent research we relate these parameters to other distance parameters.

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