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DETERMINE THE RAYLEIGH DAMPING COEFFICIENTS BASED ON NATURAL FREQUENCY FOR A SPECIFIC SYSTEM MATHEMATICALLY

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Abstract
The Rayleigh damping model is an approximation to viscous damping. It allows modeling the energy dissipation in the material due to internal friction, assuming it is proportional to the strain or deformation rate. It is common to be in the need to determine the model coefficients from experimental or assumed data. We will explore the definition to determine the Rayleigh damping coefficients. For large degrees of freedom systems, it is difficult to guess the values of Rayleigh damping coefficients α and β. There are a number of general purposes available that have the provision of providing the value of α and β for calculation of Rayleigh damping matrix for dynamic analysis of systems with multi-degree of freedom. Since a designer may not be in a position to pre-assess the same at the beginning, has no option but to assume an unrealistic constant damping ratio for all modes. Based on the present technology it is very simple to develop a spreadsheet and arrive at a rational value of α and β which produces a damping ratio sequence increasing progressively with each of the subsequent modes and one can furnish input data for the dynamic analysis. The present report outlines a procedure that ensures a rational estimate of α and β even for a system with significant degrees of freedom. The results obtained have been as a real value for The Rayleigh damping model.
Keywords: Rayleigh, damping coefficients, time-history analysis, calculation, response spectrum, FEM, directs method

1. Introduction

The formation mechanism of damping is complicated; that is, a damping matrix can be calculated by using a construction method but cannot be directly determined by identifying the material, size, and characteristics of structures [1]. Consequently, different damping matrix construction theories have been proposed [2–3]. For instance, a Rayleigh damping model is widely used because of its excellent advantages [4–5]. In this model, the damping matrix of a structure is a linear combination of mass and stiffness matrixes (1). As such, Rayleigh damping models can provide a clear physical meaning and present a convenient A Rayleigh damping matrix must be orthogonal to mode shapes (2). Consequently, decoupling dynamic equations of multiple degrees-of-freedom systems via mode superposition becomes convenient. Mode damping ratios can be directly used in single-degree-of-freedom systems (generated by decoupling) dynamic response calculation. Therefore, damping input shows enhanced accuracy and a reduced calculation scale. Rayleigh damping coefficients can be determined by the orthogonally of a damping matrix for a modal shape (6). With appropriate Rayleigh damping coefficients, the results of a dynamic response analysis of a multi-degree-of-freedom system are the same as experimental data (7). A damping model is also embedded in finite element software, and Rayleigh damping models are considered a basis for damping matrix construction commonly utilized in the seismic time history analysis of hydraulic structures. The damping force term F (t) is assumed to be proportional to the deformation rate$(x)^{\cdot}$, as seen in the general dynamic motion equation (1) for a significant degree of freedom system with inertial mass [M], damping coefficient [C] and spring stiffness [K]:

$$[M]\ddot{x} + [C]\dot{x} + [K]x = F(t) \tag{1}$$

The damping matrix of a structure is the linear combination of the mass and stiffness matrixes of a Rayleigh damping model (2):

$$[C] = \alpha[K] + \beta[M] \tag{2}$$

Where α is stiffness-proportional damping coefficient [seconds] and β is mass-proportional damping [1/seconds]. Divided the equation of motion (1) by [M] (3) and rearranging in terms of the natural frequency of oscillation $\omega_n$and the damping ratio ζ (4).

We obtain:

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{F(t)}{m} \tag{3}$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F(t)}{m} \tag{4}$$

Where: ω_n^2 and ζ as in equation (5), (6)

$$\omega_n^2 = \frac{k}{m} \tag{5}$$

$$\zeta = \frac{[C]}{c_{critic}} = \frac{[C]}{2[M]\omega_n} \tag{6}$$

Where:

ω_n =natural frequency.

ζ=Damping ratio.

C_c = critical damping coefficient.

So the frequency equation can be writing as follow

$$H_{(w)} = \frac{1}{1-\left(\omega/\omega_n\right)^2 + 2\zeta(\omega/\omega_n)} \tag{7}$$

Then we can substitute by Rayleigh damping [C] from equation (2) in equation (6) to get the value of ζ as the following equation (8), (9)

$$\zeta = \frac{1}{2\omega_n[M]}(\alpha[k] + \beta[m]) \tag{8}$$

$$\zeta = \frac{1}{2}(\alpha\omega_n + \frac{\beta}{\omega_n}) \tag{9}$$

2. RAYLEIGH DAMPING

To calculate Rayleigh damping coefficients, Chopra [1] suggested that “in dealing with practical problems, it is reasonable to select the modes of vibrations i and j with specific damping ratios to ensure that the damping ratios of all modes of vibration that contribute greatly to the dynamic response are reasonable.” Differences in the mass and stiffness of the upper and lower structures of a powerhouse remarkably create the dynamic characteristics of a powerhouse structure. The first two vibration modes often involve the relatively soft upper structure of the powerhouse, whose mode participation mass is quite smaller than that of the whole power house. The damping of these buildings under seismic actions is mainly due to various interior frictions and deformations of components and the ones between them. Therefore, mode participation mass should be considered as a key factor affecting the calculation of damping. The modes that contribute greatly to dynamic responses are found on the basis of mode participation mass. In this study, the Rayleigh damping coefficient is calculated. In traditional methods, two reference vibration modes (i- and j-order) are selected and their damping ratios ζ_i and ζ_j obtained through measurement or reliable test data estimation and their frequencies $\omega_i$and $\omega_j$are used to calculate α and β:

$$\begin{Bmatrix}\alpha \\ \beta\end{Bmatrix} = \frac{2\omega_i\omega_j}{\omega_j^2-\omega_i^2}\begin{pmatrix}\omega_j & -\omega_i \\ -\frac{1}{\omega_j} & \frac{1}{\omega_i}\end{pmatrix}\begin{Bmatrix}\zeta_i \\ \zeta_j\end{Bmatrix} \tag{10}$$

This equation can be simplified as follows when $\zeta_i = \zeta_j = \zeta$

$$\begin{Bmatrix}\alpha \\ \beta\end{Bmatrix} = \frac{2\zeta}{\omega_i-\omega_j}\begin{Bmatrix}\zeta_i \\ \zeta_j\end{Bmatrix} \tag{11}$$

Frequency can be easily and appropriately selected to determine Rayleigh damping coefficients when the degree of freedom of a structure is low. For complex structures and structures with a number of modes that contribute greatly to dynamic responses, difficulties in selecting two orders of reference frequencies to obtain reasonable Rayleigh damping coefficients α and β are encountered. If damping coefficients are chosen inappropriately, a slight difference in damping may seriously distort the calculation of the seismic response of a given structure [6–7]. Yang et al. [8] studied the application of a multi-mode-based computation method in single-layer cylindrical latticed shells because the traditional two-mode Rayleigh damping method is unsuitable. Yang et al. [8] also suggested that the multi-mode-based computation method is preferable when many dominant modes are distributed loosely and found in a wide range of frequencies under some ground motions. Jehel et al. [9] Comprehensively compared the initial structural stiffness and updated tangent stiffness of Rayleigh damping models to allow a practitioner to objectively choose the type of Rayleigh damping models that satisfy his needs and be provided with useful analytical tools for the design of these models with good control on their damping ratios during inelastic analysis. Erduran [10] evaluated the effects of a Rayleigh damping model based on the engineering demand parameters of two steel moment-resisting frame buildings. Modes that greatly influence these responses are found on the basis of mode participation mass, and Rayleigh damping coefficients are obtained. Zhiqiang Song and Chenhui Su [11].Rayleigh damping models, which combine mass and stiffness proportional components, are anchored at reduced modal frequencies, which create reasonable damping forces and floor acceleration demands for both buildings but do not suppress higher-mode effects. From this equation (9) we can see that the Rayleigh model can reproduce three cases:

3. Damping is Proportional to the Inertia

In this case, the stiffness coefficient α=0 and thus:

$$\zeta = \frac{\beta}{2\omega_n} \tag{12}$$

For a given constant value of β, it is seen that the damping is inversely proportional to the natural frequency, as shown in the illustration:

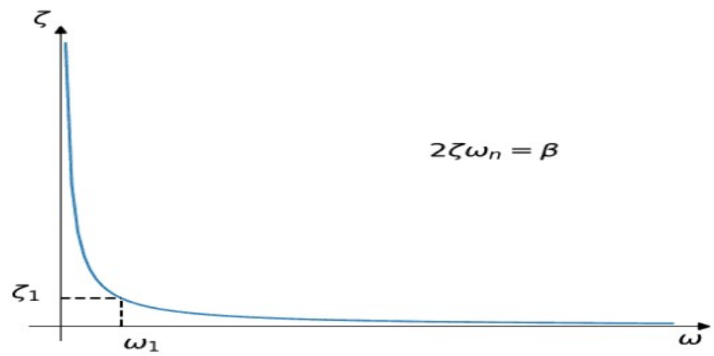

Figure 1: Schematic of damping proportional to inertia

Moreover, if one computes β from the damping ratio ζ_1 at a given natural frequency ω_1, all the natural frequencies below it will be amplified and the frequencies above it will be attenuated. The effect is more dramatic the farther the frequencies are from the reference value.

4. Damping is Proportional to the Stiffness

In this case, the mass coefficient β=0 and thus:

$$\zeta = \frac{1}{2}\alpha\omega_n \tag{13}$$

It is seen that, contrary to the first case, here the damping is directly proportional to the natural frequency:

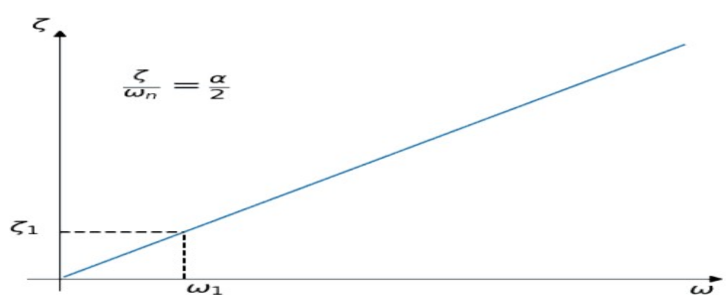

Figure 2: Schematic of damping proportional to stiffness

If one computes α from the damping ratio ζ_1 at a given natural frequency ω_1, then the natural frequencies below will be attenuated and the frequencies above will be amplified.

5. General case

In the case of using the model with two parameters, the proportionality of damping against frequency is convex:

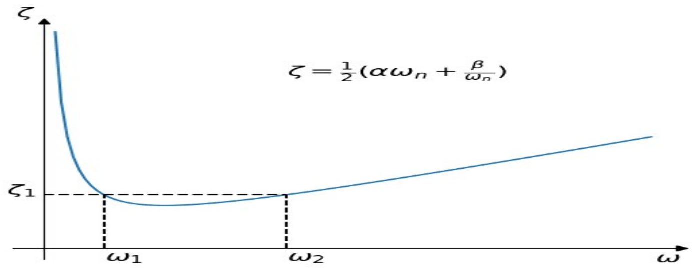

Figure 3: Schematic of the full damping model

In this case one needs two damping ratios and two natural frequencies to create a pair of equations and solve for α and β (10).

$$\begin{Bmatrix}\alpha\\ \beta\end{Bmatrix} = \frac{2\omega_i\omega_i}{\omega_j^2-\omega_i^2}\begin{pmatrix}\omega_j & -\omega_i\\ -\frac{1}{\omega_j} & \frac{1}{\omega_i}\end{pmatrix}\begin{Bmatrix}\zeta_i\\ \zeta_j\end{Bmatrix} \tag{14}$$

The model gives some flexibility on where to place the natural frequencies, but in general, frequencies too far away from the ones used in the computation will be amplified. In the particular case of using equal damping ratios for the two frequencies, it is important to note that the damping ratio will not be constant inside the range defined by the sample points, but the inner frequencies will be attenuated. That is, the inner frequencies will have a lower damping ratio.

6. Computing the Rayleigh Damping Coefficients

In the most common case, a transient response curve from the system is obtained and the damping ratio ζ_1 is determined for the lowest natural frequency ω_1 by measuring the (logarithmic) attenuation of successive peaks:

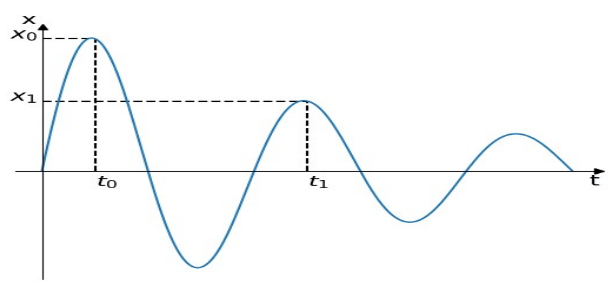

Figure 4: Determination of the damping ratio from the logarithmic decay

$$\zeta = \frac{\delta}{\sqrt{\delta^2+(2\pi)^2}} \tag{15}$$

$$\delta = \ln\frac{x_0}{x_1} \tag{16}$$

$$f = \frac{1}{T} = \frac{1}{t_1-t_0} \tag{17}$$

It is then most common to assume the case of damping proportional to the stiffness, that is, β=0, and the α stiffness coefficient is computed from:

$$\alpha = \frac{2\zeta_1}{\omega_1} = \frac{\zeta_1}{\pi f_1} \tag{18}$$

If the knowledge on the system indicates the case of damping decreasing with the frequency, then one can assume the case of damping proportional to the inertia, where α=0 and determine the mass coefficient β:

$$\beta = 2\zeta_1\omega_1 = 4\pi\zeta_1 f_1 \tag{19}$$

If there is not such test data or knowledge of the system or if one wish to apply an approximate damping ratio over a range of frequencies, and then we can use the general case and build a system of two equations:

$$\zeta_1 = \frac{1}{2}\left(\alpha\omega_1 + \frac{\beta}{\omega_1}\right) \tag{20}$$

$$\zeta_2 = \frac{1}{2}\left(\alpha\omega_2 + \frac{\beta}{\omega_2}\right) \tag{21}$$

$$\vdots$$

$$\zeta_i = \frac{1}{2}\left(\alpha\omega_i + \frac{\beta}{\omega_i}\right) \tag{22}$$

Then solve for the unknown coefficients, keeping in mind the considerations given above for the general case and the influence of the model on natural frequencies inside and outside the range of interest. That is, perhaps one wants to achieve a mean damping ratio over the range, then compensate the attenuation by modifying the input damping ratios, or by performing some least-squares approximation from more than two frequency points.

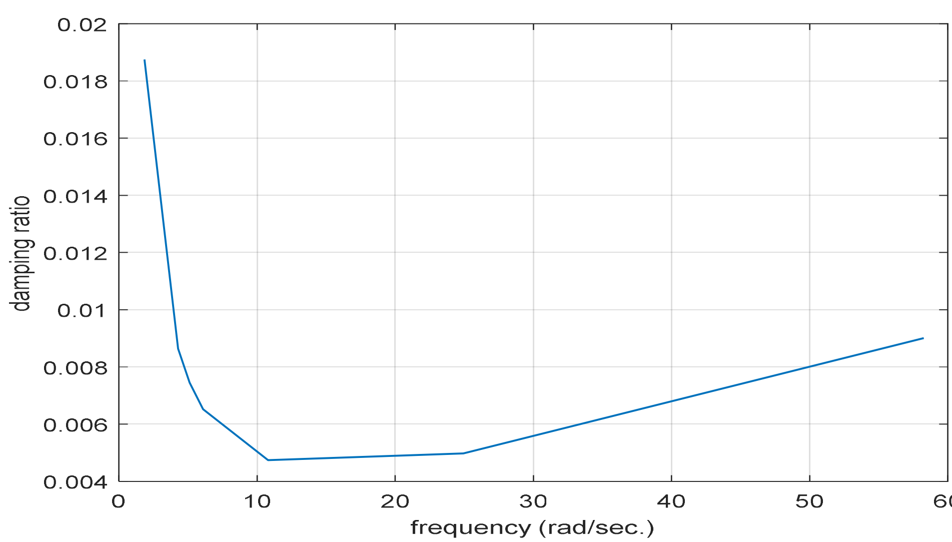

Figure 5: Variation of damping ratio with natural frequency of a system

One need not measure ζ_i, where i depending on the degree of freedom. What is relevant here is a first few modes for which there is a significant mass participation. Beyond this, results are of no practical consequence. For instance, for a steel frame of 12 degree-of-freedom if it is found that 100% mass participation occurs in the first 4 modes (assume), instead of starting with 5% constant damping for all modes, one can start with a minimum 2% damping in the first mode and define at 4th mode, ζ= 5% and this is the zone of relevance.

7. Half-power point method:-

Experimentally, we can got the damping ratio by half by half power method if the magnitude of frequency response at resonance is $H_{res} = \frac{1}{2\zeta}\ at\ \left(\frac{\omega}{\omega_n}\right) = 1$ **,** substituting by value of $H_{(w)}$ at resonance in equation (7) and take square of both sides

$$\left(\frac{1}{2\zeta}\right)^2 = \frac{1}{[1-\left(\omega/\omega_n\right)^2]^2+[2\zeta(\omega/\omega_n)]^2} \qquad (23)$$

Or:-

$$\left(\frac{\omega}{\omega_n}\right)^4 - 2(1-2\zeta^2)\left(\frac{\omega}{\omega_n}\right)^2 + (1-8\zeta^2) = 0 \qquad (24)$$

Solving equation (24) for $\left(\frac{\omega}{\omega_n}\right)^2$ so

$$\left(\frac{\omega}{\omega_n}\right)^2 = (1-2\zeta^2) \pm 2\zeta\sqrt{1-\zeta^2} \qquad (25)$$

Assume $\zeta \leq 1$ so we can rewrite equation (25) as follow:

$$\left(\frac{\omega}{\omega_n}\right)^2 = 1 \pm 2\zeta \qquad (26)$$

Let ω_1, ω_2 is the roots of equation (26) and $\omega_2 > \omega_1$ then the equation become

$$4\zeta = \frac{\omega_2^2-\omega_1^2}{\omega_n^2} \cong 2\left(\frac{\omega_2-\omega_1}{\omega_n}\right) \qquad (27)$$

The damping ratio can be written by rearranged equation (27) as follow

$$\zeta = \frac{\omega_2-\omega_1}{2\omega_n} = \frac{f_2-f_1}{2f_n} \qquad (28)$$

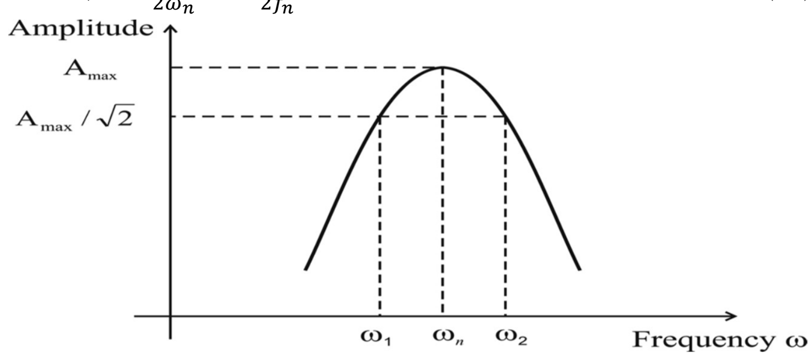

Figure 6: Half-power bandwidth method

The two algebraic equation can be solved to determine the coefficient α, β if both mods are assumed to have the same damping ratio ζ then the

$$\alpha = \zeta\frac{2\omega_1\omega_2}{\omega_1+\omega_2} \qquad (29)$$

$$\beta = \zeta\frac{2}{\omega_1+\omega_2} \qquad (30)$$

And from the bandwidth method and equation (28)

$$\zeta = \frac{\Delta\omega}{2\omega_2} \qquad (31)$$

Where:- $\Delta\omega = \omega_1 + \omega_2$

And substituting from equation (31) into equation (29), (30) to get the value damping constants (α, β) if we assume ω_n at the middle distance in between ω_2, ω_1 from the figure (6)

$$\omega_n = \frac{\omega_2-\omega_1}{2} + \omega_1 = \frac{\omega_2+\omega_1}{2} \qquad (31)$$

So

$$\zeta = \frac{\omega_2-\omega_1}{\omega_2+\omega_1} \qquad (32)$$

Substituting from equation (32) into equation (29), (30)

$$\alpha = \frac{2\omega_1\omega_2(}{(\omega_1+\omega_2)(}\frac{\omega_2-\omega_{1)}}{\omega_2+\omega_1)} \tag{33}$$

$$\beta = \frac{2}{(\omega_1+\omega_2)}\frac{(\omega_2-\omega_1)}{(\omega_2+\omega_1)} \tag{34}$$

Now,

Substituting from equation (34), (35) into equation (2)

$$[C] = \frac{2\omega_1\omega_2(}{(\omega_1+\omega_2)(}\frac{\omega_2-\omega_{1)}}{\omega_2+\omega_1)}[M] + \frac{2}{(\omega_1+\omega_2)}\frac{(\omega_2-\omega_1)}{(\omega_2+\omega_1)}[K] \tag{36}$$

Where:

$[C]$ the damping coefficient for structure. Or

$$[C] = \frac{2(}{(\omega_1+\omega_2)(}\frac{\omega_2-\omega_{1)}}{\omega_2+\omega_1)}\left[\omega_1\omega_2[M] + [K]\right] \tag{37}$$

So from equation (37) we can get the value of [C] Instead of frequency values and matrix [M], [K] as above .the suitable value of α and β from equation (33) and equation (34). The results of this study obtained (α_1&β) are the proportional damping constants that have suitable value for our case is 0.0687 and 2.89e-4 respectively.

8. Conclusion

A number of general purpose available Finite Element Analysis package have the provision of providing the value of α and β for calculation of Rayleigh damping matrix for dynamic analysis of systems with multi-degree of freedom. Since a designer is not in a position to pre-assess the same at the beginning, he has no option but to assume a constant damping ratio for all modes, which is unrealistic. Based on the present technique it is very simple to develop a spreadsheet and arrive at a rational value of α and β developing a damping ratio sequence which increases progressively with each of the subsequent modes and can furnish an input data for the dynamic analysis. The value furnished by this method gives a more realistic picture for the behavior of the structure under dynamic loading than the presumptive damping ratio constant for all modes. The results of this study obtained (α_1&β) are the proportional damping constants that have suitable value for our case is 0.0687 and 2.89e-4 respectively.

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