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ISOMORPHISM ON SEMIPRIME (-1,1) RINGS

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ABSTRACT: In this paper we show that a 2-divisible (-1,1) ring satisfies the identities (x, (y, x, y)) = 0, (x, (y, z), x) = 0, (x, (y, z), w) = ((y, z), w, x) = (w, x, (y, z)). using these identities, we prove that a semiprime 2-divisible (-1,1) ring is isomorphic to a subdirect sum of a semiprime alternative and a semiprime commutative ring. Also, we prove that a prime 2-divisible (-1,1) ring R is alternative or commutative.

KEY WORDS: Right alternative rings, (-1,1) Rings, Semiprime rings, Isomorphism, semiprime (-1,1) ring, prime (-1,1) rings.

INTRODUCTION

Block [1] defined the completely alternative elements. Thedy [2] studied rings with completely alternative commutators. He proved that the ring R is alternative or commutative by assuming the conditions (x, (y, x, y)) = 0, (x, (y, z), x) = 0, (x, (y, z), w) = ((y, z), w, x) = (w, x, (y, z)) and 3-divisibility. Using these identities and isomorphism property we prove that a 2-divisible prime (-1,1) ring R is alternative or commutative.

Throughout this paper R will denote a (-1,1) ring. A non associative ring R is defined to be (-1,1) if the following two identities hold:

$$(x, y, y) = 0, \text{ i.e. } (x, y, z) + (x, z, y) = 0 \quad (1)$$
$$(x, y, z) + (y, z, x) + (z, x, y) = 0 \quad (2)$$

∀ x,y,z in R.

PRELIMINARIES:

R is said to be prime if A and B are ideals of R and such that AB = 0, then either A = 0 (or) B = 0. R is said to be semiprime if for any ideals A of R, $A^2 = 0$ implies A =0. R is called k-divisible if kx = 0 implies x = 0, x ϵ R and k is a natural number.

The commutative center U of R is the set of all elements u in R such that (u, R) = 0. An alternative ring R is a ring in which (xx) y = x (xy) and y (xx) = (yx) x for all x, y in R. These equations are known as the the left and right alternative laws respectively. The nucleus N of R is the set of all elements n in R such that (n,R,R) = (R,R,n) = (R,n,R) = 0.

We know that in any ring the following identities

$$(wx,y,z) – (w,xy,z) + (w,x,yz) = w(x,y,z) + (w,x,y)z \quad (3)$$

$$(xy,z) – x(y,z) – (x,z)y – (x,y,z) + (x,z,y) – (z,x,y) = 0. \quad (4)$$

Hold.

In a (-1,1) ring R we have the following identities [2, 3]:

$$(x,(y,y,x)) = 0 \quad (5)$$

$$(x,(y,z)) + (y, (z,x)) +(z,(x,y)) = 0 \quad (6)$$

$$(w,(x,y,z)) = 0 \quad (7)$$

Using identities of (-1,1) rings we prove the the following main results.

Main results:

Lemma 1: - Let R be a (-1,1) ring. Then (x, (y, x, y)) = 0 and (x,(y,z),x) = 0.

Proof:- From (5) , we have (x,(y,x,y)) = 0

Consider the equation (5), (y,(x,x,y) = 0.

By replacing y by y + (a, b) in the above the equation then we obtain ((a,b), (x,x,y)) = -(y,(x,x,(a,b))) (8)

Substituting y = (R,R,R) an arbitary associator in (6) and apply (7) then we have ((R,R,R), (z,x)) =0 (9)

From (8) and (9), we get (y,(x,x,(a,b))) =0.

This implies that (x,x,(a,b)) ∊ U.

Arguing as in the proof of theorem 6 in [4], we have (x,x,(y,z)) = 0.

By using right alternative, we obtain

$$(x,(y,z),x) = 0. \quad (10)$$

This completes the proof of the Lemma ◻

Lemma 2:- In a (-1,1) ring R,

$(x,(y,z),w) = ((y,z),w,x) = (w,x,(y,z))$.

Proof:- Substituting $(y,z) = v$ in (10) we have

$$(x,v,x) = 0. \quad (11)$$

By linearizing $(x,v,x) = 0$ and using (1) we get,

$$(x,v,y) = -(y,v,x) = (y,x,v) \quad (12)$$

from this we have

$$(v,y,x) = (x,v,y). \quad (13)$$

From (12) & (13) we obtain

$(x,v,y) = (v,y,x) = (y,x,v)$

This completes the proof of Lemma. □

To prove next lemmas we define the following subsets in a (-1,1) ring R .

$V = \{v/v \in R, (x,v,x) = 0 \text{ and } (v,y,x) = (y,x,v) = (x,v,y) \text{ for all } x, y \in R\}$

and $W = \{v/v \in V, R\,v + v\,R \subset R\}$. V is the alternative nucleus of R which consists of all completely alternative elements of R and a subgroup of R containing the nucleus of R.

If U is an ideal of R, then we denote $U^{\perp}$ by the sum of all ideals S of R such that SU = US = 0. Here $U^{\perp}$ is the maximal ideal annihilating U from both sides.

Lemma 3:- Let U be an ideal of R contained in V.

Then $U^{\perp} = \{p/p \in R, pU = Up = 0\}$.

Proof:- We have only to show that $\{p/p \in R, pU = Up = 0\}$, is an ideal of R.

Let $p \in R$ such that $pU = Up = 0$.

Then we have for $v \in U$ and $x \in R$

$(x,p,v) = -(x,v,p) = -(p,x,v) = (p,v,x) = 0$

Hence $(xp)U = (px)U = U(px) = U(xp) = 0$. □

Lemma 4:- The alternative nucleus V is an alternative sub algebra of R such that $(xw,x,v) = (w,x,v)x$ and $(v,x,wx) = x(v,x,w)$ for $x \in R$ and $v, w \in V$.

Proof: - Let $x \in R$ and $v, w \in V$. Then from (3), (1) and definition of V we have

$$(x,vw,x) = 0 \quad (14)$$

From (11) and (1) , we get (x,x,v) = 0 and (v,x,x) = 0.

From (3) and (1), we get

(x,x,vw + wv) = (x,xv,w) + (x,xw,v)

= - (xv,x,w) – (xw,x,v)

= - (x,vx,w) + (x,v,xw) – (x,wx,v) + (x,w,xv)

= (w,vx,x) + (v,wx,x) – (x,xw,v) – (x,xv,w)

= - (x,x,vw + wv)

Therefore 2(x,x,vw + wv) = 0, since R is 2 - divisible we have (x,x,vw + wv) = 0.

From this we obtain (xv,w,x) + (xw,v,x) = 0 and (vx,w,x) + (wx,v,x) = 0.

Now first we see that (xw,x,v) – (w,x,v)x

= (xw,x,v) – (wx,v,x) + (w,xv,x) – (w,x,vx)

= - (xw + wx,v,x) – (xv + vx,w,x) =0.

Similarly we get (v,x,wx) – x(v,x,w)

= (v,x,wx) – (xv,x,w) + (x,vx,w) – (x,v,xw)

= (xw + wx,v,x) + (xv + vx,w,x) = 0.

It remains to show that (vw,y,x) = (y,x,vw) = (x,vw,y) for v,w$\in$V.

Linerizing (14), we have (x,vw,y) = - (y,vw,x) = (y,x,vw).

From this, we get (vw,y,x) = (x,vw,y).

Hence (vw,y,x) = (y,x,vw) = (x,vw,y).

Therefore V is a sub algebra and trivially V is alternative. □

Lemma 5:- If R satisfies (xw,x,y) = (w,x,y)x and (y,x,wx) = x(y,x,w) for x, y $\in$ R, w $\in$ W, then (y,y,x) $\in$ $U^{\perp}$ for any ideal of R contained in V.

Proof: - w(y,y,x) = (wy,y,x) – (w,y^2,x) + (w,y,yx) – (w,y,y)x

= y(y,x,w) – (y^2,x,w) + (y,yx,w)

= - (y,y,x)w

= y(x,w,y) – (y^2,x,w) + (yx,w,y)

= - (y,wy,x) + y(y,w,x) = 0.

Hence (y,y,x) $\in$ $U^{\perp}$ by Lemma 3. □

Lemma 6 :- If R is a 2-divisible (-1,1) ring, then W is an ideal of R contained in V such that (R,V) + (R,R,V) + (R,V,R) + (V,R,R) $\subset$ W.

Proof:- Let $v \in V$ and $x, y \in R$. From (4), (2), we have

$$(xy,v) + (vy,x) - v(y,x) - x(y,v) = (x,y,v) + (v,y,x) = 0.$$

Since V is sub algebra by lemma 4 and $(R,R) \subset V$ we get $(R,V) \subset W$.

From (4), we have $(x,y,v) = (xy,v) - (x,v)y - x(y,v) \in V$.

Hence $(x,y,v) = (v,x,y) \in V$ and consequently W is an ideal of R.

Therefore $(x,y,v) = (xy,v) - (x,v)y - x(y,v) \in W$ implies

$$(x,y,v) = (y,v,x) = (v,x,y) \in W \quad \text{or}$$

$$(R,R,V) + (R,V,R) + (V,R,R) \subset W.$$

This completes the proof of the lemma. □

Corollary:- In a (-1,1) ring R, $(x,y)^2 \in W$ for $x, y \in R$.

Proof:- Since $(x,y) \in V$ we have modulo W that $(x,y)^2 \equiv (x,y(x,y))$.

From (4), we obtain that $y(x,y) = (yx,y) - (y,x,y)$. From $(x,(y,x,y)) = 0$, commuting this equation with x we get $(x,y)^2 \equiv (x,(yx,y))$.

Hence $(x,y)^2 \in W$ by Lemma 6. □

Theorem 1:- A semiprime 2-divisible (-1,1) ring R is isomorphic to a sub-direct sum of a semiprime alternative and a semiprime commutative ring.

Proof:- From $(W^{\perp\perp} \cap W^{\perp})^2 \subset W^{\perp\perp} W^{\perp} = 0$ we get $W^{\perp\perp} \cap W^{\perp} = 0$.

Hence the canonical homomorphism $R \rightarrow R/W^{\perp\perp} \oplus R/W^{\perp}$ is an isomorphism.

By lemma 5 $R/W^{\perp}$ is an alternative ring. From the lemma 6 and corollary we have that the ideal T generated by (x,y) and W satisfies $T^2 \subset W$. The ideal $T \cap W^{\perp}$ squares to zero. Hence $T \cap W^{\perp} = 0$ and consequently $T W^{\perp} + W^{\perp}T = 0$ or $T \subset W^{\perp\perp}$. This proves that $(x,y) \in W^{\perp\perp}$ and $R/W^{\perp\perp}$ is commutative. It remains to show that $R/W^{\perp\perp}$ and $R/W^{\perp}$ are semiprime rings.

Let I be an ideal of R such that $I^2 \subset W^{\perp\perp}$. Then $(I \cap W^{\perp})^2 \subset W^{\perp\perp} \cap W^{\perp} = 0$ shows that $I \cap W^{\perp} = 0$. Hence, $I W^{\perp} + W^{\perp} I \subset I \cap W^{\perp} = 0$ or $I \subset W^{\perp\perp}$. This shows that $R/W^{\perp\perp}$ is semiprime. Similarly, we get that $R/W^{\perp}$ is semiprime. □

Theorem 2: - A prime 2-divisible (-1,1) ring R is alternative or commutative.

Proof: - A ring R is called prime if the product of any two non-zero ideals of R is non-zero. Clearly a prime ring is semiprime.

Let R be a subring of the direct sum of an alternative ring A and a commutative ring C. Any alternator i.e., element of type (y,y,x) or (x,y,y) lies in C and any commutator (x,y) lies in A.

Therefore, the ideal I of R generated by all alternators of R and the ideal J of R generated by all commutators of R satisfy I J = 0. If R is prime then either I = 0 and R is alternative or J = 0 and R is commutative. Hence from theorem 1 we have that a prime (-1,1) ring R is alternative or commutative. □

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