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CONSTRUCTING ORTHOGONALITY AND DUAL CODE APPLICATION

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Abstract:
We aim to construct an orthogonal matrix Q that transform a given column of A-call it$x$ into the corresponding column of R-call it y. For this we found $\|x\| = \|Qx\| = \|y\|$.We extended the notation of orthogonal projection that we explained it we also discussed only projection onto a single vector (or, equivalently,the one dimensional subspace spanned by that vector).We saw that we can find the analogous formulas for projection onto a plane in $IR.^3$The orthogonality is becoming an important mathematical tools in linear algebra, there are many ways of constructing new cods from old ones.Here ,we consider one of the most.
Keywords : Orthogonality , orthogonal matrix , Dual Code Application , orthogonal projection .

1. Introduction:

The modified QR factorizationand its applications are very important tools in mathematics, when we are dealing with these tools number of similar applications of common results can be found .This results can be specified as the modified QR factorizationthat are independent from a particular specification; but sometimes such modules are not so simple: A general module that can satisfy different purposes is not trivial.Moreover, the more complicated modules often differ very slightly from application to application.Immediately one asks: Does the modified QR factorizationanswer these quantity estimates; involving tools such sum ability methods.

2. Orthogonality in R^n:

Definition (2.1):

A set of vectors $(v_1, v_2, \dots, v_k)$ in R^n is called an orthogonal set if all pairs of distinct vector in the set are orthogonal. That is, if $V_i . V_j = 0$ whenever $i \neq j$for$i, j = 1, 2, \dots, k$. [4]

Example (2. 2):

Show that (v_1, v_2, v_3) is an orthogonal set in R^3 if.

$$v_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Solution:

We must show that every pair of vector from this set is orthogonal. This is true, since.

$$v_1 . v_2 = 2.0 + 1.1 + -1.1 = 0$$
$$v_2 . v_3 = 0.1 + 1.(-1) + 1.1 = 0$$
$$v_1 . v_3 = 2.1 + 1.(-1) + -1.1 = 0$$

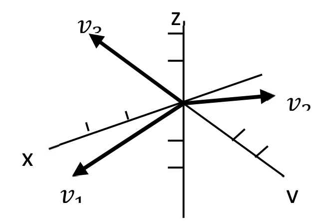


Fig N0. (1) : Orthogonal Set in R^3

Theorem (2. 3):

If the vectors in set $(v_1, v_2, \dots, v_k)$ in R^n are mutually orthogonal and nonzero then that set is linearly independent.

Proof

If $c_1, \dots, c_k$ are scalars such that

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$$

Then

$(c_1v_1 + \ c_2v_2 + \ \dots + c_kv_k). v_i$

$= 0. v_i = 0$

Show, since v_i is non zero, thatc_i is zero. [5]

Definition (2. 4):

An orthogonal basis for a subspace W of R^n is a basis of W that is an orthogonal set.

Example (2. 5):

Findan orthogonal basis for the subspace w, of R^3given by

$$W = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x - y + 2z = 0\right\}$$

Solution:

The subspace W is a plane through the origin in R^3. Fromthe equation of the plane, we have $x = y - 2z$, so Wconsists of vectors of the form

$$\begin{bmatrix} y - 2z \\ y \\ z \end{bmatrix} = y\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

It follows that $u = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $v = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ are a basis for W, but they are not orthogonal it suffices to find another nonzero vector in W that is orthogonal to either one of these.

Suppose $W = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is a vector in w that is orthogonal to u.

Then $x - y + 2z = 0$

Since w is in the plane W. since $u. w = 0$

We also have $x + y = 0$

Solving the linear system

$$\begin{aligned} x - y + 2z &= 0 \\ x + y \qquad &= 0 \end{aligned}$$

We find that $x = \ -z$ and $y = z$

Thus any nonzero vector w of the form $W = \begin{bmatrix} -z \\ z \\ z \end{bmatrix}$ will do. To be specific, we could take $W = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$. It is easy to check that (u, w) is an orthogonal set in W and hence an orthogonal basis for w, since$\dim W = 2$.

Definition (2. 6):

A set of vector in R^n is an orthonormal set if it is an orthogonal set of unit vectors. An orthonormal basis for a subspace W of R^n is a basis of W that is an orthonormal set.

Remark (2. 7):

If $s = (q_1, \dots, q_k)$ is an orthonormal set of vectors, then $q_i . q_j = 0$ for $i \neq j$and $\|q_i\| = 1$. The fact that each q_i is a unit vector is equivalent to $q_i . q_j = 1$.

It follows that we can summarize the statement that S is orthonormal as:[10]

$$q_i . q_j = \begin{cases} 0 & if\ i \neq j \\ 1 & if\ i = j \end{cases}$$

Example (2. 8):

Show that $S = (q_1, q_2)$ is an orthonormal set in R^3 if

$q_1 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}andq_2 = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$

Solution:

We check that

$$q_1 . q_2 = 1/\sqrt{18} - 2/\sqrt{18} + 1/\sqrt{18} = 0$$

$$q_1 . q_1 = 1/3 + 1/3 + 1/3 = 1$$

$q_2 . q_2 = 1/6 + 4/6 + 1/6 = 1$ #

If we have an orthogonal set. We can easily obtain an orthonomal set from it. We simply normalize vector.

Example (2. 9):

Construct an orthonomal basis for R^3 from the vectors.

$v_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Solution:

Since we already know that v_1, v_2 and v_3 an orthogonal basis, we normalize them to get

$$q_1 = \frac{1}{\|v_1\|} v_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$$

$$q_2 = \frac{1}{\|v_2\|} v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$q_3 = \frac{1}{\|v_3\|} v_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

Then (q_1, q_2, q_3) is an orthonomal basis for R^3.

Theorem (2. 10):

The column of an$m \times n$ matrix Q form an orthonomal set if and only if:

$$Q^T Q = I_n$$

Proof

We need to show that

$$(Q^T Q)_{ij} = \begin{cases} 0 & if\ i \neq j \\ 1 & if\ i = j \end{cases}$$

Let q_i denote the ith column of Q (and, hence the ith row of the Q^T).

Since the(i, j) entry of $Q^T Q$ is the dot product of the ith row of Q^T and the jth column of Q, it follows that

$$(Q^T Q)_{ij} = q_i . q_j (*)$$

By the definition of matrix multiplication.

Now the columns Q form an orthonormalset if and only if:

$$q_i . q_j = \begin{cases} 0 & if\ i \neq j \\ 1 & if\ i = j \end{cases}$$

Which by equation (*) holds if and only if

$$(Q^T Q)_{ij} = \begin{cases} 0 & if\ i \neq j \\ 1 & if\ i = j \end{cases}$$

Definition (2. 11):

An $n \times n$ matrix Q whose column form an orthonormal set is called an orthogonal matrix.

Theorem (2. 12):

A square matrix Q is orthogonal if and only if $Q^{-1} = Q^T$.

Example (2. 13):

Show that the following matrices are orthogonal and find their inverses:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\ and\ B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Solution:

The columns of A are just the standard basis vectors for R^3. Which are clearlyorthonormal

Hence, A is orthogonal

$$A^{-1} = A^T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

For B, we check directly that

$$B^T B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$B^T B = I$

Therefore, B is orthogonal, the theorem (2. 12)and

$$B^{-1} = B^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Theorem (2. 14):

Let Q be an$n \times n$ matrix. The following statements are equivalent:

i. Q is orthogonal.

ii. $\|Qx\| = \|X\|$for every x in R^n.

iii. $Qx.Qy = x.y$for every x and y in R^n.

Proof

We will prove that $(i) \Longrightarrow (iii) \Longrightarrow (ii) \Longrightarrow (i)$.

To do so, we will need to make use of the fact that if x and y are (column) vectors in R^n, then $x.y = x^T y$.

$(i) \Longrightarrow (iii)$ Assume that Q is orthogonal. Then $Q^T Q = I$ and we have

$$\begin{aligned} Qx.Qy &= (Qx)^T Qy \\ &= x^T Q^T Qy \\ &= x^T I y \\ &= x^T y \\ &= x.y \end{aligned}$$

$(iii) \Longrightarrow (ii)$ Assume that $Qx.Qy = x.y$ for every x and $y$in R^n.

Then, taking $y = x$ we have$Qx.Qx = x.x$ so

$$\begin{aligned} \|Qx\| &= \sqrt{Qx.Qx} \\ &= \sqrt{x.x} \\ &= \|x\| \end{aligned}$$

$(ii) \Longrightarrow (i)$ Assume that property (ii) holds and let q_i denoted the ith column of Q.

$$\begin{aligned} x.y &= \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) \\ &= \frac{1}{4}(\|Q(x+y)\|^2 - \|Q(x-y)\|^2) \\ &= \frac{1}{4}(\|Qx+Qy\|^2 - \|Qx-Qy\|^2) \\ &= Qx.Qy \end{aligned}$$

for all$x$ and y in R^n.

Now if e_i is the ith standard basis vector then $q_i = Q\, e_i$.

Consequently,

$$\begin{aligned} q_i.q_j &= Qe_i.Qe_j \\ &= e_i.e_j \\ &= \begin{cases} 0 & if\ i \neq j \\ 1 & if\ i = j \end{cases} \end{aligned}$$

Thus, the column of Q form an orthonormal set, so Q is an orthogonal matrix.

Theorem (2. 15):

If Q is an orthogonal matrix, then its rows form an orthnormal set.

Proof

By theorem (2. 12). We know that $Q^{-1} = Q^T$

Therefore:

$$\begin{aligned} (Q^T)^{-1} &= (Q^{-1})^{-1} \\ &= Q \\ &= (Q^T)^T \end{aligned}$$

So Q^T is an orthogonal matrix.
Thus the column of $Q^T -$. Which are just rows of Q- form an orthonormal set.

3. Orthogonal Complement and Orthogonal Projection :

Definition (3.1):

Let W be a subspace of R^n, we say that a vector v in R^n is orthogonal to W if v is orthogonal to every vector in W. the set of all vectors that are orthogonal to w is called the orthogonal complement of W, denoten$w^{\perp}$. That is [11]

$$w^{\perp} = \{v \ in \ R^n : v.w = 0; for \ all \ w \ is \ W\}$$

Theorem (3.2):

Let A be an $m \times n$ matrix. Then the orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T.

$\left(row\ (A)\right)^{\perp} = null\ (A) \quad and \quad \left(col(A)\right)^{\perp} = null(A^T)$

Proof

If x is a vector in R^n, then x is in $\left(row\ (A)\right)^{\perp}$if and only if x is orthogonal to every row of A. but this is true if and only if $Ax = 0$, which is equivalent to x being in null (A), so we have established the first identity.

To prove the second identity, we simply replace A by A^T and use the fact that $row(A^T) = col(A)$.

Definition (3.3):

Let W be a subspace of R^n and let $\{u_1, \dots, u_k\}$ be an orthogonal basis for W. for any vector in R^n, the orthogonal projection of v onto w is defined as:

$$proj_w(v) = \left(\frac{u_1.v}{u_1.u_1}\right)u_1 + \cdots + \left(\frac{u_k.v}{u_k.u_k}\right)u_k$$

The component of v orthogonal to w is the vector

$$prep_w(v) = v - proj_w(v)$$

Theorem (3.4):

Let w be a subspace of R^n and let v be a vector in R^n. Then there are unique vectors w in W and $w^{\perp}$ in$W^{\perp}$such that

$$v = w + w^{\perp}$$

Proof

We need to show two things: that such a decomposition exists and that it is unique.

To show existence, we choose an orthogonal basis $\{u_1, \dots, u_k\}$ for w. let $w = proj_w(v)$ and let $w^{\perp} = perp_w(v)$; Then

$$\begin{aligned} w + w^{\perp} &= proj_w(v) + perp_w(v) \\ &= proj_w(v) + \left(v - perp_w(v)\right) \\ &= v \end{aligned}$$

Clearly;
$w = proj_w(v)$is in w.
Since it is a linear combination of the basis vectors $(u_1, \dots, u_k)$.
To show that $w^{\perp}$is in $W^{\perp}$, it is enough to show that $w^{\perp}$ is orthogonal to each of the basis vectors u_i, we compute

$$u_i . w^{\perp} = u_i . perp_w(v)$$
$$= u_i . \left(v - proj_w(v)\right)$$
$$= u_i . \left(v - \frac{u_1 . v}{u_1 . u_1}\right) u_1 - \cdots \left(\frac{u_k . v}{u_k . u_k}\right) u_k$$
$$= u_i . v - \left(\frac{u_1 . v}{u_1 . u_1}\right)(u_i u_1) - \cdots - \left(\frac{u_i . v}{u_i . u_i}\right)(u_i u)_i - \cdots - \left(\frac{u_k . v}{u_k . u_k}\right)(u_i . u_k)$$
$$= u_i . v - 0 - \cdots - \left(\frac{u_i . v}{u_i . u_i}\right) u_i u_i - \cdots - 0$$
$$= u_i . v - u_i . v = 0$$

Since $u_i . u_j = 0 \; for \; j \neq i$.

This proves that $w^{\perp}$ is in $W^{\perp}$ and completes the existence part of the proof.

To show that uniqueness of this decomposition, let's suppose we haveanotherdecomposition

$$v = w_1 + w_1^{\perp}$$

Where w_1 is in W and $w_1^{\perp}$ is in $W^{\perp}$; then

$$w + w^{\perp} = w_1 + w_1^{\perp}$$

So..

$$w - w_1 = w_1^{\perp} - w^{\perp}$$

But since $w - w_1$ is in W and $w_1^{\perp} - w^{\perp}$ is in $w^{\perp}$ (because these are subspaces). We know that this common vector is in $W \cap W^{\perp} = \{0\}$. Thus $w - w_1 = w_1^{\perp} - w^{\perp} = 0$, so $w_1 = w$and$w_1^{\perp} = w_1$.

4. The Gram-Schmidt Process and the QR factorization :

We present a simple method for constructing an orthogonal (or orthonormal) basis for any subspace of R^n. This method will then lead us to one of the most useful of all matrix factorizations.

Example (4.1):

Let $w = span\ (x_1, x_2)$, where

$x_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}andx_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

Construct an orthogonal basis for W.

Solution:

Starting with x_1, we get a second vector that is orthogonal to it by taking the component of x_2 orthogonal to x_1

Algebraically, we set $v_1 = x_1$, so

$$v_2 = perp_{x_1}(x_2) = x_2 - proj_{x_1}(x_2)$$
$$= x_2 - \left(\frac{x_1 . x_2}{x_1 . x_1}\right) x_1$$
$$= \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \left(-\frac{2}{2}\right)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Then $[v_1, v_2]$ is an orthogonal set of vectors in W.

Hence, $[v_1, v_2]$ is a linearly independent set and therefore a basis for Wsincedim $w = 2$.

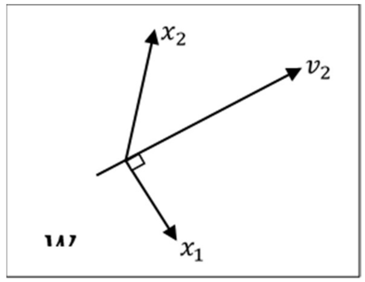


Fig No. (2) : Orthogonal Basis for W

Theorem (4.2):

Let $\{x_1, \dots, x_2\}$ be a basis for a subspace W of R^n and define the following.

$v_1 = x_1 w_1 = span\ (x_1)$

$v_2 = x_2 - \left(\frac{v_1 . x_2}{v_1 . v_1}\right) v_1 w_2 = span\ (x_1, x_2)$

$v_3 = x_3 - \left(\frac{v_1 . x_3}{v_1 . v_1}\right) v_1 - \left(\frac{v_2 . x_3}{v_2 . v_2}\right) v_2 w_3 = span\ (x_1, x_2, x_3)$

$$\vdots$$

$v_k = x_k - \left(\frac{v_1 . x_k}{v_1 . v_1}\right) v_1 - \left(\frac{v_2 . x_k}{v_2 . v_2}\right) v_2 - \cdots - \left(\frac{v_{k-1} . x_k}{v_{k-1} . v_{k-1}}\right) v_{k-1} w_k = span\ (x_1, \dots, x_k)$

Then for each $i = 1, \dots, k, \{v_1, \dots, v_i\}$ is an orthogonal basis for W_i. in particular, $\{v_1, \dots, v_k\}$ is an orthogonal basis for w.

Proof

We will prove by induction that, for each $i = 1, \dots, k, \{v_1, \dots, v_i\}$ is an orthogonal basis for W_i.

Since$v_1 = x_1$, clearly$\{v_1\}$ is an (orthogonal) basis for $W_1 = span(x_1)$.

Now assume that, for some $i < k, \{v_1, \dots, v_i\}$ is an orthogonal basis for w_i. Then

$$v_{i+1} = x_{i+1} - \left(\frac{v_1 . x_{i+1}}{v_1 . v_1}\right) v_1 - \left(\frac{v_2 . x_{i+1}}{v_2 . v_2}\right) v_2 - \cdots - \left(\frac{v_i . x_{i+1}}{v_i . v_i}\right) v_i$$

By induction hypothesis, $\{v_1, \dots, v_i\}$ is an orthogonal basis for $span\ (x_1, \dots, x_i) = w_i$. Hence,

$$v_{i+1} = x_{i+1} - proj_{w_i}(x_{i+1})$$
$$= perp_{w_i}(x_{i+1})$$

So, by the orthogonal decomposition theorem, v_{i+1} is orthogonal to w_i. By definition, $v_1, \dots, v_i$ are linear combination of $x_1, \dots, x_i$ and, hence, are in w_i therefore $\{v_1, \dots, v_{i+1}\}$ is an orthogonal set of vector in w_{i+1}. Moreover, $v_{i+1} \neq 0$, since otherwise

$$X_{i+1} = proj_{w_i}(x_{i+1})$$

Which is turn implies that x_{i+1} is in W_i .But this is impossible, since $w_i = span\ (x_1, \dots, x_i)$ and $\{x_1, \dots, x_{i+1}\}$ is linearly independent.

We conclude that $\{v_1, \dots, v_{i+1}\}$ is a set of i+1 linearly independent vectors in w_{i+1}. Consequently $\{v_1, \dots, v_{i+1}\}$ is a basis for w_{i+1}, since

$$\dim w_{i+1} = i + 1$$

Example (4.3):

Apply the Gram-Schmidt process to construct an orthonormal basis for subspace $w = span\ (x_1, x_2, x_3)$of R^4, where

$$x_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} x_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} x_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

Solution:

First we note that (x_1, x_2, x_3) is a linearly independent set, so it forms a basis for w.

We begin by setting $v_1 = x_1$. Next we compute the component of x_2 orthogonal to $w_1 = span\ (v_1)$

$$v_2 = perp_{w_1}(x_2) = x_2 - \left(\frac{v_1 . x_2}{v_1 . v_1}\right) v_1$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{4} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3/2 \\ 3/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

For hand calculations, it is a good idea to "scale"v_2 at this point to eliminate fractions when we are finished, we can rescale the orthogonal set we are constructing to obtain an orthogonal set; thus, we can replace each v_i by any convenient scalar multiple without affecting the final result.

Accordingly, we replace v_2 by

$$\grave{v}_2 = 2v_2 = \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

We now find the component of x_3 orthogonal to:

$$w_2 = span(x_1, x_2) = span(v_1, v_2) = span(v_1, \grave{v}_2)$$

Using the orthogonal basis $(v_1, \grave{v}_2)$

$$v_3 = perp_{w_2}(x_3) = x_3 - \left(\frac{v_1 . x_3}{v_1 . v_1}\right) v_1 - \left(\frac{\acute{v}_2 . x_3}{\acute{v}_2 . \acute{v}_2}\right) \acute{v}_2$$

$$= \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix} - \left(\frac{1}{4}\right) \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} - \left(\frac{15}{20}\right) \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \\ 1 \end{bmatrix}$$

Again , we rescale and use $\acute{v}_3 = 2v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$

We now have an orthogonal basis $(v_1, \acute{v}_2, \acute{v}_3)$ for w, to obtain an orthonormal basis, we normalize each vector

$$q_1 = \left(\frac{1}{\|v_1\|}\right) \quad v_1 = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

$$q_2 = \left(\frac{1}{\|\acute{v}_2\|}\right)\acute{v}_2 = \frac{1}{2\sqrt{5}}\begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2\sqrt{5} \\ 3/2\sqrt{5} \\ 1/2\sqrt{5} \\ 1/2\sqrt{5} \end{bmatrix} = \begin{bmatrix} 3\sqrt{5}/10 \\ 3\sqrt{5}/10 \\ \sqrt{5}/10 \\ \sqrt{5}/10 \end{bmatrix}$$

$$q_3 = \left(\frac{1}{\|\acute{v}_3\|}\right)\acute{v}_3 = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{6} \\ 0 \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} = \begin{bmatrix} -\sqrt{6}/6 \\ 0 \\ \sqrt{6}/6 \\ \sqrt{6}/3 \end{bmatrix}$$

Then (q_1, q_2, q_3) is an orthonormal basis for w.

Definition (4.4):

If A is an $m \times n$ matrix with linearly independent columns (requiring that $m \geq n$) then applying the Gram-Shamidt process to these columns yields a very useful factorization of A into the product of a matrix Q with orthonormal columns and an upper triangular matrix R. this is the QR factorization, and it has applications to the numerical approximation of eigenvalues, which we explore at the end of this section.

To see how the QR factorization arises let $a_1, \dots, a_n$ be the (linearly independent) column of A. andlet $q_1, \dots, q_n$ be the orthonormal vectors obtained by applying the Gram-Schmidt process to A with normalizations from theorem (5-21) we know that, for each $i = 1, \dots, n$

$$W_i = span(a_1, \dots, a_i) = span\ (q_1, \dots, q_i)$$

Therefore, there are scalars,$r_{1i}, r_{2i}, \dots, r_{ii}$ such that

$$a_i = r_{1i}q_1 + r_{2i}q_2 + \cdots + r_{ii}q_i \qquad for\ i = 1, \dots, n$$

That is

$$\begin{gathered} a_1 = r_{11}q_1 \\ a_2 = r_{12}q_1 + r_{22}q_2 \\ \vdots \\ a_n = r_{1n}q_1 + r_{2n}q_2 + \cdots + r_{nn}q_n \end{gathered}$$

Which can be written in matrix form as

$$A = [a_1 a_2 \dots a_n] = [q_1 q_2 \dots q_n]\begin{bmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ 0 & r_{22} & \dots & r_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & r_{nn} \end{bmatrix}$$

$$A = QR$$

Clearly,

the matrix Q has orthonormal columns. It is also the case that the diagonal entries of R are all non-zero.

To see this, observe that if $r_{ii} = 0$, then $a_i$is a linear combination of $q_1, \dots, q_{i-1}$ and, hence, is in W_{i-1}. But then a_i would be a linear combination of $a_1, \dots, a_{i-1}$, which is impossible, since

$a_1, \dots, a_i$ are linearly independent. We conclude that $r_{ii} \neq 0$for $i = 1, \dots, n$. Since R is upper triangular, it follows that it must be invertible.

Theorem (4.5):

Let A be an$m \times n$ matrix with linearly independent columns. Then A can be factored as $A = QR$, where Q is an $m \times n$ matrix with orthonormal columns and R is an invertible upper triangular matrix.

Proof

To prove this, suppose that A is an$m \times n$ matrix that has a QR factorization then, since R is invertible, we have

$$Q = AR^{-1}$$

Hence,$rank\ (Q) = ranK\ (A)$

But $rank\ (Q) = n$

Since its columns are orthonormal and therefore, linearly independent so $rank\ (A) = n$ too, and consequently the columns of A are linearly independent by the fundamental theorem.

The QR factorization can be extended to arbitrary matrices in a slightly modified form. If A $\times n$, it is possible to find a sequence of orthogonal matrices $Q_1, \dots, Q_{m-1}$ such that

$$Q_{m-1}, \dots, Q_2 Q_1 A$$

In an upper triangular $m \times n$ matrix R then.$A = QR$

Where $Q = (Q_{m-1}, \dots, Q_2 Q_1)$ is an orthongonal matrix.[6]

Example (4.6):

Find a QR factorization of $A = \begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

Solution:

The columns of A are just the vectors from example (4.3). The orthonormal basis for col(A) produced by the Gram-Schmidt process was;

$$q_1 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}, q_2 = \begin{bmatrix} 3\sqrt{5}/10 \\ 3\sqrt{5}/10 \\ \sqrt{5}/10 \\ \sqrt{5}/10 \end{bmatrix}, q_3 = \begin{bmatrix} -\sqrt{6}/6 \\ 0 \\ \sqrt{6}/6 \\ \sqrt{6}/3 \end{bmatrix}$$

So:

$$Q = [q_1 q_2 q_3]$$

$$Q = \begin{bmatrix} 1/2 & 3\sqrt{5}/10 & -\sqrt{6}/6 \\ -1/2 & 3\sqrt{5}/10 & 0 \\ -1/2 & \sqrt{5}/10 & \sqrt{6}/6 \\ 1/2 & \sqrt{5}/10 & \sqrt{6}/3 \end{bmatrix}$$

From theorem (4.5)

$$A = QR$$

For some upper triangular matrix R. to find R, we use the fact that Q has orthonormal columns and, hence,

$$Q^T Q = I$$

Therefore,

$$Q^T A = Q^T QR$$
$$Q^T A = IR$$
$$Q^T A = R$$

We compute: $R = Q^T A$

$$= \begin{bmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ 3\sqrt{5}/10 & 3\sqrt{5}/10 & \sqrt{5}/10 & \sqrt{5}/10 \\ -\sqrt{6}/6 & 0 & \sqrt{6}/6 & \sqrt{6}/3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1/2 \\ 0 & \sqrt{5} & 3\sqrt{5}/2 \\ 0 & 0 & \sqrt{6}/2 \end{bmatrix}$$

5. The Modified QR Factorization

When the matrix A does not have linearly independent columns, the Gram-Schmidt process as we have stated it does not work and so cannot be used to develop a generalized QR factorization of A. there is a modification of the Gram-Schmidt process that can be used, but instead we will explore a method that converts A into upper triangular form one column at a time, using a sequence of orthogonal matrices. The method is analogous to that of Lu factorization in which the matrix L is formed using sequence of elementary matrices.

The first thing we need is the "orthogonal analogue" of an elementary matrix, that is, we need to know how to construct an orthogonal matrix Q that will transform a given column of A-call it x –into the corresponding column of R-call it y.

By theorem (4.2), it will be necessary that

$\|x\| = \|QX\| = \|y\|$.

Fig No.(3) suggests away to proceed.

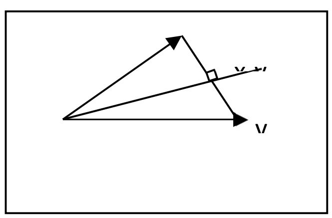


Fig (3) : Modification of the Gram-Schmidt Process

we can reflect x in a line perpendicular to $x - y$. If

$u = \left(\frac{1}{\|x - y\|}\right)(x - y) = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$

In the unit vector in the direction of $x - y$, then $u^{\perp} = \begin{bmatrix} -d_2 \\ d_1 \end{bmatrix}$ is orthogonal to u.

We can generalize the definition of Q as follows. If u is any unit vector in R^n.

We define an$n \times n$ matrix Q as:

$$Q = I - 2uu^T$$

Such a matrix is called a householder matrix (or an elementary reflector).

6.Dual Codes:

There are many ways of constructing new codes from old ones. In this section, we consider one of the most important of these.

First, we need to generalize the concepts of a generator and parity check matrix for a code.

Definition (6.1):

For $n > m$, an $n \times m$ matrix G and an $(n-k) \times n$ matrix P (with entries in Z_2)are a generator matrix and parity check matrix, respectively, for an (n, k) binary code C if the following condition are all satisfied:

i. The column of G are linearly independent.
ii. The rows of P are linearly independent.
iii. PG=0

Notice that property (iii) implies that every column V of G satisfies $Pv = 0$ and so is a code vector in C. also, a vector y is in C if and only if it obtained from the generator matrix as $y = Gu$ for some vector u in Z_2^n. In other words, C is the column space of G.

To understand the relationship between different generator matrices for the same code, we only need to recall that, just as elementary row operations do not affect the row space of a matrix , elementary column operation do not affect the column space. For matrix over z_2, there are only two relevant operation: interchange two columns (C1) and add one column to another column $C2$.

Similarly, elementary row operations preserve the linear independence of the rows of P. moreover, if E is an elementary matrix and c is a code vector,

Then,

$$(EP)_C = E(Pc) = E0 = 0$$

It follows that EP is also a parity check matrix for C. thus, any parity check matrix can be converted into another one by means of a sequence of row operations; interchange two rows$(R1)$ and add one row to another row (R_2). We are interested in showing that any generator or parity check matrix can be brought into standard form there is one other definition we need. We will call two cods $C_1$and C_2 equivalent if there is a permutation matrix M such that

$$\{Mx\colon x\ in\ C_1\} = C_2$$

In other words, if we permute the entries of the vectors in C_1 (all in the same way) we can obtain C_2. For example:

$$c_1 = \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$$

and,

$$c_2 = \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$$

are equivalent via the permutation matrix

$$M = \begin{bmatrix}0 & 0 & 1\\1 & 0 & 0\\0 & 1 & 0\end{bmatrix}$$

permuting the entries of code vectors corresponds to permuting the row of generator matrix and permuting the columns of a parity check matrix for the code.

We can bring any generator matrix for code into standard form by mean of operation C_1, C_2 and R1. If $R1$ has not been used, then we have the same code;if R1 has been used, then we have an equivalent code. We can bring any parity check matrix for a code into standard form by mean of operation $R1, R2$ andC1. If C1 has not been used, then we have the same code, if C1 has been used, then we have an equivalent code .The following example illustrate these points.

Example (6.2):

Bring the generator matrix

$$G = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Into standard form and find an associated parity check matrix.

Solution:

We can bring the generator matrix G into standard form as follows:.

$$G = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} I \\ \cdots \\ A \end{bmatrix} = \acute{G}$$

Hence,

$$A = [1 \quad 0]$$

So, $P = [A \ \vdots \ I] = [1 \ 0 \ 1]$

Definition (6.3):

Let C be a set of code vectors in Z_2^n. The orthogonal complement of C is called the dual code of C and is denoted $C^{\perp}$. That is,

$$C^{\perp} = \{x \ in \ Z_2^n : c.x = 0 \ for \ all \ c \ in \ C \ \}$$

Theorem (6.4) :

If C is an(n, k) binary code with generator matrix G and parity check matrix P, then $C^{\perp}$ is an $(n, n-k)$ binary code such that:

i. $C^T$is a parity check. Matrix for $C^{\perp}$.
ii. $P^T$is a generator matrix for $C^{\perp}$.

Proof

By definition, G is an $n \times k$ matrix with linearly independent column, p is an $(n-k) \times n$ matrix with linearly independent rows, and $PG = 0$. Therefore, the rows of G^T are linearly independent, the columns of P^T are linearly independent and$G^T P^T = (GP)^T = 0^T = 0$

This shows that G^T is a parity check matrix for $C^{\perp}$ and P^T is a generator matrix for $C^{\perp}$. Since

$P^T$is$n \times (n-k)$

$C^{\perp}$is an $(n, n-k)$ code.

Example (6.5):

Find generator and parity check matrices for the dual code $C^{\perp}$ as follow

$$P = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Solution:

$$C^{\perp} = P^T = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

This matrix is in standard form with

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

So a parity check matrix for $C^{\perp}$ is

$$P^T = [A \vdots I] = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Theorem (6.6):

If C is a self dual code, then:

i. Every vector in C has even weight.

ii. 1 is in C.

Proof

i. A vector x in z_2^n has even weight if and only if $w(x) = 0$ in z_2

But $w(x) = x.x = 0$

Since c is self dual.

ii. Using property (i) we have $1.x$

$= w(x) = 0$

In z_2 for all x in C.

This mean that 1 is orthogonal to every vector in C.

So 1 is in $C^{\perp} = C$, as required.

Definition (6.7):

Quadratic form

A quadratic form in n variables is a function $f: R^n \to R$ of the form

$$f(x) = x^T A x$$

Where A is a symmetric $n \times n$matrix and x is in R^n. We refer to A as the matrix associated with f.

Example (6.8):

What is the quadratic form with associated matrix

$$A = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$$

Solution:

If $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Then

$$f(x) = x^T A x$$

$$= [x_1 x_2] \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 2x_1^2 + 5x_2^2 - 6x_1 x_2$$

Generalization: we can expand a quadratic form in n variables $x^T A x$ as follows:

$$x^T A x = a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2 + \sum 2a_{ij}x_i x_j$$

Thus, if$i \neq j$, the coefficient of x_j, x_i is $2a_{ij}$.

Theorem (6.9):

Every quadratic form can be diagonalized specifically, if A is the $n \times n$ symmetric matrix associated with the quadratic form $x^T A x$ and if Q is an orthogonal matrix such that $Q^T A Q = D$ is a diagonal matrix, then the change of variable $x = Qy$ transforms the quadratic form $x^T Ax$ into the quadratic form $y^T Dy$, which has no cross-product terms. If the eigenvalues of A are $\lambda_1, \ldots, \lambda_n$ and $y = [y_1, \ldots., y_n]^T$, then

$$\begin{aligned} x^T Ax &= y^T Dy \\ &= \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2 \end{aligned}$$

Theorem (6.10):

Let $f(x) = x^T Ax$ be a quadratic form with associated $n \times n$ symmetric matrix A.

Let the eigenvalues of A be $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then the following are true, subject to the constraint $\|x\| = 1$:

i. $\lambda_1 \geq f(x) \geq \lambda_n$
ii. The maximum value of f(x) is λ_1, and it occurs when x is a unit eigenvector corresponding to λ_1.
iii. The minimum value of f(x) is λ_n and it occurs when x is a unit eigenvector corresponding to λ_n.

Proof

As usual, we begin by orthogonally diagonalizing A. accordingly , let Q be an orthogonal matrix such that $Q^T AQ$ is the diagonal matrix

$$D = \begin{bmatrix} \lambda_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \lambda_n \end{bmatrix}$$

Then, by the principal Axes theorem, the change of variable $x = Qy$ gives

$$x^T Ax = y^T Dy$$

Now note that:

$$y = Q^T x$$

Implies that

$$\begin{aligned} y^T y &= (Q^T x)^T (Q^T x) \\ &= x^T (Q^T)^T Q^T x \\ &= x^T Q Q^T x \\ &= x^T x \end{aligned}$$

Since $Q^T = Q^{-1}$

Hence using $x.x = x^T x$, we see that

$$\begin{aligned} \|y\| &= \sqrt{y^T y} \\ &= \sqrt{x^T x} \\ &= \|x\| \\ &= 1 \end{aligned}$$

Thus, if x is a unitvector, so is the corresponding y, and the values of $x^T Ax$ and $y^T Dy$ are the same.

i. to prove (i), we observe that if $y = [y_1, \ldots, y_n]^T$,

then

$$\begin{aligned} f(x) &= x^T Ax = y^T Dy \\ &= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \\ &\leq \lambda_1 y_1^2 + \lambda_1 y_2^2 + \cdots + \lambda_1 y_n^2 \\ &= \lambda_1 (y_1^2 + y_2^2 + \cdots + y_n^2) \end{aligned}$$

$$= \lambda_1 \|y\|^2$$
$$= \lambda_1$$

Thus, $f(x) \leq \lambda_1$ for all x such that $\|x\| = 1$

ii. ifq_1 is a unit eigenvector corresponding to λ_1. Then

$$Aq_1 = \lambda_1 q_1$$

and

$$f(q_1) = q_1^T A q_1$$
$$= q_1^T \lambda_1 q_1$$
$$= \lambda_1 (q_1^T q_1)$$
$$= \lambda_1$$

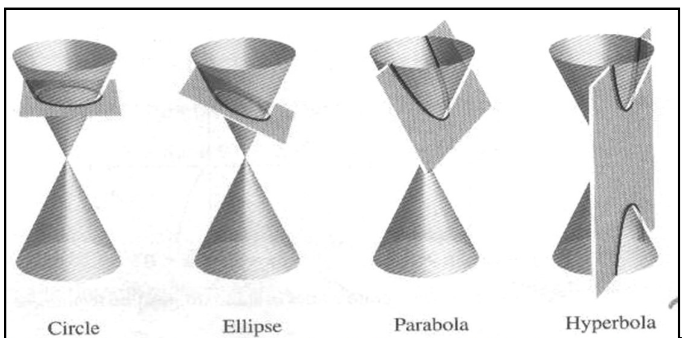
This shows that the quadratic form actually take on the valueλ_1 and so, by property (i), it is the maximum value of $f(x)$ and it occurs when $x = q_1$.

Definition (6.11):

The general form of a quadratic equation in two variables x and y is

$$ax^2 + by^2 + cxy + dx + ey + f = 0$$

Where at least one of a, b and c is nonzero. The graphs of such quadratic equations are called conic section (or conics), since they can be obtained by taking cross sections of a (double) cone. (i.e, slicing it with a plane). The most important of the conic sections are the ellipses (with circle a special case), hyperbolas, and parabolas. These are called the nondegenerate conics. See fig No. (4)



It also possible for a cross section of a cone to result in a single point, a straight line, or a pair of lines. These are called degenerate conics.

The graph of a non degenerate conics is said to be in standard position relative to the coordinate axes if its equation can be expressed in one of the forms in fig No. (5).

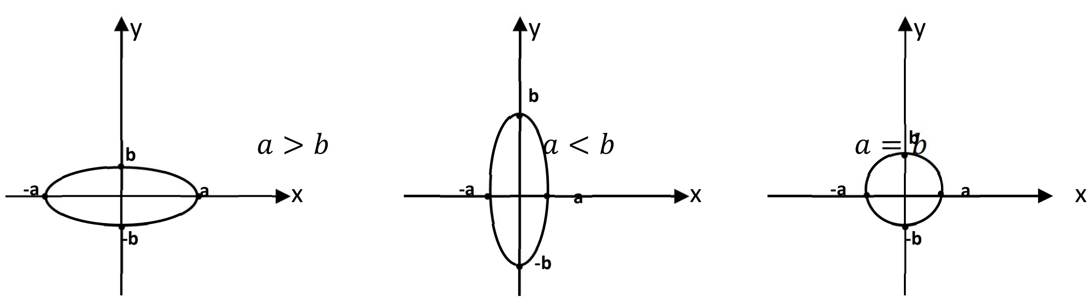


Fig No. (5) : Non Degenerate Conics in Standard Position

Definition (6.12):

Where at least one of $a, b, \dots, f$ is nonzero. The graph of such a quadratic equation is called a quadric surface.

Example (6.13):

Identify the quadric surface whose equation is

$$5x^2 + 11y^2 + 2z^2 + 16xy + 20xy - 4yz = 36$$

Solution:

The equation can be written in matrix form as $x^T A x = 36$, where

$$A = \begin{bmatrix} 5 & 8 & 10 \\ 8 & 11 & -2 \\ 10 & -2 & 2 \end{bmatrix}$$

We find the eigenvalues of A to be 18,9 and -9, with corresponding orthogonal eigen vectors

$$\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \quad and \quad \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

Respectively, we normalize them to obtain

$$q_1 = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}, \qquad q_2 = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix} \qquad and \qquad q_3 = \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}$$

And form the orthogonal matrix

$$Q = [q_1 q_2 q_3]$$

$$Q = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

Not that in order for Q to be the matrix of a rotation, we require

$$\det Q = 1$$

Which is true in this case. (Otherwise, $\det Q = -1$and swappingtwo columns changes the sign of the determinant). [50]

Therefore

$$Q^T A Q = D$$

$$= \begin{bmatrix} 18 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -9 \end{bmatrix}$$

and with the change of variable $x = Q\acute{x}$

We get

$$x^T A x = (\acute{x}) D \acute{x}$$

$$= 36$$

So

$$18(\acute{x})^2 + 9(\acute{y})^2 - 9(\acute{z})^2 = 36$$

Or

$$\frac{(\acute{x})^2}{2}+\frac{(\acute{y})^2}{4}-\frac{(\acute{z})^2}{4}=1$$

Conclusion:

The study dealt with the modified QR factorization and its applications and we took the model study (orthogonal) and we came to the results of which is access to the relationship through the modified QR factorization and we focused on the applications link questioner to make the study of study come as an application for the modified QR factorization so they can be the beginning of advanced study in concept the modified QR factorization and its applications.

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