Journal of Data Acquisition and Processing
ISSN: 1004-9037
https://sjcjycl.cn/
DOI: 10.5281/zenodo.98549495

MATHEMATICAL SYSTEM APPLICATION SOLUTIONS USING RUNGE-KUTTA METHOD

Athraa N Albukhuttar and Aseel A Hussain
Department of Mathematics, College of Education for Girls, University of Al-Kufa, Najaf - Iraq

Abstract: In this paper, Runge- Kutta of the second and fourth order are applied to solve systems of linear differential equations of first order or single equation of second order which is converted to system of linear differential equations of first order. Moreover, the results of the solutions are compared, and the relative errors are computed that gave approvals in the application of the methods studied.
Keyword: Runge- kutta method; Approximate solution; Systems of linear differential equations

1 Introduction

Systems of differential equations appear in many fields of science, including finance, computing, and mechanics. Solving a system of differential equations means finding the values of the variables that make each equation true, where it can be written in the form of a vector or arrays and solved by the usual classical methods. Integral transforms one of the methods are used to solve systems that characterized by high accuracy and speed such as Laplace, Noval, Shehu and Elzaki transform...etc [1,2,3,5].

In addition, different numerical methods are used for solving these systems of differential equations to get approximate solutions of mathematical problems[4,7], where the use of direct methods to solve these systems requires great computational effort, so many researchers go to solve the use of iterative methods (approximate methods) that do not calculate the direct solution, but start with an approximate value, for instance the Jacobi iterative method, the Gauss-Seidel iterative method and Taylor method ...etc[6,11,12]. The researchers worked on developing these methods and solving them in more accurate and effective ways, and using them to solve a lot of applications in other sciences.

Rung -Kutta is one of the important and accurate numerical methods for solving ordinary differential equations [8,9].This method depends on the initial values of the system, where the authors applied it in many mathematical applications[10,13,14].

In this work, Some applications that is expressed in a linear system of ordinary differential equations of the first order or a single differential equation of the second order which is converted into a linear system of the first order are solved, such as the salt deposition system in the tank, the spring equation and the charge equation in the electrical circuit

2-Preliminaries.

In this section ,we introduced some preliminaries which need in the following work.

2.1) System of linear differential equations:

An $m \times m$ system of first order linear ODEs is a set of m differential equations involving m unknown functions $s_1, \cdots, s_n$ and has the form:

$$
\begin{aligned}
\frac{ds_1}{dt} &= a_{11}s_1(t) + a_{12}s_2(t) + \cdots + a_{1m}s_m(t) + g_1(t) \\
\frac{ds_2}{dt} &= a_{21}s_1(t) + a_{22}s_2(t) + \cdots + a_{2m}s_m(t) + g_2(t) \\
&\vdots \\
\frac{ds_m}{dt} &= a_{m1}s_1(t) + a_{m2}s_2 + \cdots + a_{mm}s_m(t) + g_m(t)
\end{aligned}
\qquad \cdots (1)
$$

If each of the functions $g_j(t)$ is identically zero, then the system (1) is said to be homogeneous; otherwise it is nonhomogeneous, $a_{11}, \cdots, a_{mm}$ are constan coefficients.The vector valued function

$$\frac{dS}{dt} = S'(t) = \begin{bmatrix} s_1'(t) \\ s_2'(t) \\ \vdots \\ s_m'(t) \end{bmatrix}, \quad A= \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1m} \\ a_{21} & a_{22} \cdots & a_{2m} \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{mm} \end{bmatrix} \quad and\ G(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_m(t) \end{bmatrix}$$

The left side of (1) are the components of the vector$S'(t)$ while the right side of (1) are the components of the vector $A(t)S(t) + g(t)$, and we can write equation (1) in the concise form

$$S'(t) = A(t)\, S(t) + G(T)$$

Moreover, if $, s_1(t), s_2(t), \cdots, s_n(t)$ satisfy the initial conditions $s_1(t_0)$=s_1^0 , $s_2(t_0)$=$s_2^0, \cdots, s_m(t_0) = s_m^0$ then S(t) satisfies the initial value problem

$$S'(t) = A(t)\, S(t) + G(t) \quad , \quad S(t_0) = S^0 \quad , \quad \text{where} \quad S^0 = \begin{bmatrix} s_1^0 \\ s_2^0 \\ \vdots \\ s_m^0 \end{bmatrix}$$

2.2) Conversion of Higher Order Equations to First Order Systems

Every m^{th} order differential equation for the single variable $\mathcal{Y}$

$\mathcal{Y}^{(m)}(t) = f(t, \mathcal{Y}, \mathcal{Y}' \cdots \mathcal{Y}^{(m-1)})$

can be converted into a system of m first-order equations for the variables

$$\begin{pmatrix} S_1(t) \\ S_2(t) \\ \vdots \\ S_m(t) \end{pmatrix}' = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ a_{1m} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \\ \vdots \\ S_m \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ g(t) \end{pmatrix} \quad .$$

(2-3) Numerical Solutions for Systems of differential equations with initial values .

In this section, we derivative Runge- Kutta methods for system in dimension two m=2.

(2.3.1) Runge-kutta method of Second-order

We generalize the method of Runge- Kutta from the second order to solve the system of initial value problem of two differential equations (1), by applying them to the axes, the binary solution of Υ_{i}, Z_i and it has the iterative form:

$$y_{i+1} = y_i + \frac{1}{2}(I_1 + I_2)$$

$$Z_{i+1} = Z_i + \frac{1}{2}(L_1 + L_2) \qquad \cdots (2.1)$$

$$I_1 = \mathfrak{h} \cdot f_1(t_{i,} y_{i,} z_i)$$
$$L_1 = \mathfrak{h} \cdot f_2(t_{i,} y_{i,} z_i)$$
$$I_2 = \mathfrak{h} \cdot f_1(t_i + \mathfrak{h}, y_i + I_1, z_i + L_1)$$
$$L_2 = \mathfrak{h} \cdot f_2(t_i + \mathfrak{h}, y_i + I_1, z_i + L_1) \quad \cdots (2.2)$$

i=0,1,2,⋯ with initial condition $\Upsilon(t_0) = \Upsilon_0$, $Z(t_0) = z_0$.

(2.3.2) Runge-kutta method of fourth-order

We suffice with generalizing the traditional method of Runge- Kutta from the fourth order to solve the initial value:

$\Upsilon'(t) = f_1$ (t, $\Upsilon, z(t)$)

$Z'(t) = f_2$ (t, $\Upsilon, z(t)$)

And get the formula.

$$y_{i+1} = y_i + \frac{1}{6}(I_{1+} 2I_2 + 2I_3 + I_4)$$
$$Z_{i+1} = Z_i + \frac{1}{6}(L_{1+} 2L_2 + 2L_3 + L_4) \quad \cdots (3.1)$$

$$I_1 = \mathfrak{h} \cdot f_1(t_{i,} y_{i,} z_i)$$
$$L_1 = \mathfrak{h} \cdot f_2(t_{i,} y_{i,} z_i)$$
$$I_2 = \mathfrak{h} \cdot f_1(t_i + \frac{\mathfrak{h}}{2}, y_i + \frac{I1}{2}, z_i + \frac{L_1}{2})$$
$$L_2 = \mathfrak{h} \cdot f_2(t_i + \frac{\mathfrak{h}}{2}, y_i + \frac{I1}{2}, z_i + \frac{L_1}{2})$$
$$I_3 = \mathfrak{h} \cdot f_1(t_i + \frac{\mathfrak{h}}{2}, y_i + \frac{I_2}{2}, z_i + \frac{L_2}{2})$$
$$L_3 = \mathfrak{h} \cdot f_2(t_i + \frac{\mathfrak{h}}{2}, y_i + \frac{I_2}{2}, z_i + \frac{L_2}{2})$$
$$I_4 = \mathfrak{h} \cdot f_1(t_i + \mathfrak{h}, y_i + I_3, z_i + L_3)$$
$$L_4 = \mathfrak{h} \cdot f_2(t_i + \mathfrak{h}, y_i + I_3, z_i + L_3) \quad \cdots (3.2)$$

i=0,1,2,⋯ with initial condition $\Upsilon(t_0) = \Upsilon_0$, $Z(t_0) = z_0$.

(2.4) The Numerical Solution of m th-Order Initial Value Problem

The $m^{th} -$ order differential equation is written in regular form as follows:

$$F\left(t, y, y', \cdots, y^{(m-1)}, y^{(m)}\right), a \le t \le b$$

or explicitly:

$$y^{(m)} = f\left(t, y, y', \cdots, y^{(m-1)}\right) \quad \cdots (4.1)$$

The general solution to the differential equation of order m includes a set of constants represintian conditional at the starting point, which are called the initial conditions of the differential equation(4.1) .

$$y^{(p)} = y_0^{(p)} \quad , p = 0,1,2,\cdots, m-1 \qquad (4.2)$$

The differential equation (4.1) with initial condition (4.2) can be simply converted into a sentence it is composed of one of the differential equations of the first order as follows:

$$y' = y_1' = y_2, \qquad y_1(a)=y_0'$$
$$y_2' = y_3, \qquad y_2(a)=y_0'$$
$$y_3' = y_4, \qquad y_3(a)=y_0''$$
$$\vdots$$
$$y_{m-1}' = y_m, \qquad y_m(\text{a})=y_0^{(m-2)}$$
$$y_m' = f(t, y_1, \cdots, y_m), \qquad y_m(a) = y_0^{(m-1)} \qquad \cdots (5)$$

Thus, we will solve the initial value problem in the set of equations (6) instead of the initial value problem (4.1) (4.2) because they are equivalent, we will explain the numerical solution to this issue by confining it to order *m=2* then our question becomes:

$$y''(t) = f\big(t, y(t), y'(t)\big) \qquad , \qquad \cdots (6.1)$$

with initial condition:

$$y(a) = y_0, \qquad y'(a) = y_0' \qquad , \qquad \cdots (6.2)$$

To the problem of the initial value in the sentence of the two differential equations:

$$y'(t) = z(t), \qquad y(a) = y_0 \qquad , \quad \cdots (7.1)$$
$$y''(t) = z'(t) = f\big(t, y(t), z(t)\big), \quad z(a) = z_0 = y_0' \qquad , \quad \cdots (7.2)$$

(2.5) Remark: The error calculated in these application s is the relative error symbolized E_R(8,9).

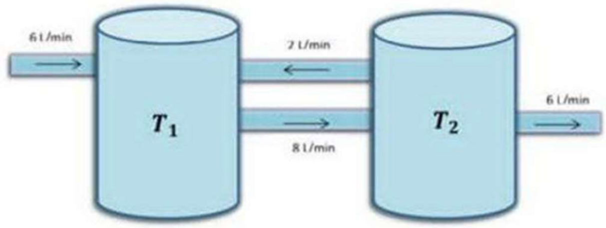
$$e_\Upsilon = |\Upsilon_{exact} - \Upsilon_{approximate}| \qquad \cdots (8)$$
$$E_R = \frac{|\xi_\Upsilon|}{\Upsilon_{exact}} \times 100\% \qquad \cdots (9)$$

3- Application

In this section,the amount of salt in Tanks can be expressed as system of differential equations of first order which solved by Runge-Kutta.Also vibrating with electric equations can be converted to system of first order which solved by Runge-Kutta as in following problem.

(3-1)Tank Application :

suppose we have two tanks, each containing 24 liters of sea water , and also containing cylinders connecting them . Water is pushed under pressure from the first Tank T_1 at a speed of 8 L/ m to the second tank T_2 , and with less pressure,water pushed from the second tank T_2 to the first tank T_1 at aspeed of 2 L/m .In addition , the first tank T_1 is supplied with fresh water at aspeed of 6 while the water is emptied from the second tank at an equal speed (64 m).Each of two tank contains an amount of salt as shown in figure(1), so if we want calculate the mass of salt present in each tank at a specific time , we express the problem in following system:



Figure(1) Tank application

Let g_1(t)=Amount of salt in Tank T_1
and g_2(t)=Amount of salt in Tank T_2
Recall that the rate of change of the salt,

$$g_i' = \boldsymbol{Rate\ In - Rate\ Out}$$

=(Concentration In)(Flow Rate In)-(Concentration Out)(Flow Rate Out)

$$V_1 = 24 + t(6 + 2 - 8) = 24$$
$$V_2 = 24 + t(8 - 2 - 6) = 24$$

$$g_1'(t) = (0)(6) + (\tfrac{g_2}{24})\,(2) - (\tfrac{g_1}{24})(8)$$

$$g_2'(t) = \left(\frac{g_1}{24}\right)(8) - \left(\frac{g_2}{24}\right)(2) - (\frac{g_2}{24})\,(6)$$

Which simplifies to the system:

$g_1'(t) = -\frac{1}{3}\, g_1 + \frac{1}{12}$, $g_1(0)=1$

$g_2'(t) = \frac{1}{3}\, g_1 - \frac{1}{3}\, g_2$, $g_2(0)=6$, $\cdots (10)$

the exact solution of system (10) is $e^{\frac{-t}{2}}\begin{bmatrix}-1\\2\end{bmatrix} + 2e^{\frac{-t}{6}}\begin{bmatrix}1\\2\end{bmatrix}$.

We solve this system of ordinary differential equation using Runge-Kutta for second(2.1) (2,2), and fourth order (3.1),(3.2) on intervel [0,0.5] and ɧ $= 0.1$ as in Table(1):

Table(1) The solution of Tank Application by second and fourth order of RungeKuttta

t_n	Exacte S. y_{i+1} z_{i+1}	RK2 y_{i+1} z_{i+1}	Error (y_{i+1}) Error (z_{i+1})	RK4 y_{i+1} z_{i+1}	Error (y_{i+1}) Error (z_{i+1})
0.1	1.015713 5.836344	1.015644 5.836388	6.793257544-E4 7.538966175-E5	1.01571348058 5.83634466949	2.520395803-E8 8.926820295-E9
0.2	1.029594 5.678539	1.029558 5.678623	3.496523873-E4 1.479253731-E4	1.0295947780 5.6785392479	4.759153874-E8 1.761016272-E8
0.3	1.041750 5.526333	1.041699 5.526454	4.895608351-E4 2.189516991-E4	1.0417508656 5.5263336649	6.623464575-E8 2.551420325-E8
0.4	1.052283 5.379489	1.052218 5.379642	6.177045529-E4 2.844136311-E4	1.0522832081 5.3794894642	8.362767607-E8 3.346042441-E8
0.5	1.061288 5.237779	1.061211 5.237962	7.255335027-E4 3.493847297-E4	1.0612880357 5.2377792460	9.799413117-E8 4.085701035-E8

(3-2)Vibrating spring.

If we put body has mass M = 1 kg at the end of a spring then the motion of spring given damping take force at $\mathfrak{P}$= 5 .Also , suppose that constant of spring has value $\mathcal{K}$= 4 and the function of externnal has the form f(t)=cost. If the object is released from rest at $\frac{3}{34}$ cm below its equilibrium. Determine the displacement of the object at any time t:

with initial condition $\chi(0) = \frac{3}{34}$ cm $\chi'(0) = 0$.

This problem can be expressed as an equation :-

$$\frac{d^2\chi}{dt^2} + 5\frac{d\chi}{dt} + 4\chi = \text{cost} \qquad \cdots (11)$$

the exact solutionof equation (11) is: $-\frac{5}{102}\, e^{-t} + \frac{5}{102}\, e^{-4t} + \frac{3}{34}\cos(t) + \frac{5}{34}\sin(t)$

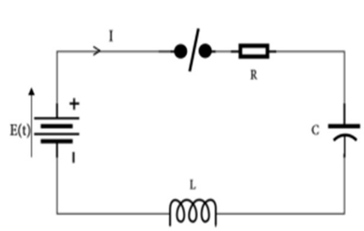
To solve the system of equation by use the Matlap problem with initial condition $\chi(0) = \frac{3}{34}$ cm , $\chi'(0) = 0$ on an interval [0,0.5] and ђ =0.1 After converting equation(11)to system of first order and using equation(3.2) as shown in Table (2):

Table(2) The solution of vibrating spring problem by second and fourth order of Runge-Kutta

t_n	Exacte S. y_{i+1} z_{i+1}	RK2 y_{i+1} z_{i+1}	Error(y_{i+1}) Error(z_{i+1})	RK4 y_{i+1} z_{i+1}	Error (y_{i+1}) Error (z_{i+1})
0.1	0.0909799	0.0914709 0.0482796	5.396796435-E2	0.0909843 0.0504170	4.814249122-E4
0.2	0.0975846	0.09823716 0.07564642	6.687120714-E2	0.0975907 0.0786034	6.250986324-E4
0.3	0.1062030	0.1068462 08856342	6.056326092-E2	0.1062094 0,0196466	6.026195117-E4
0.4	0.1155755	0.1161282 0.09146346	4.782155388-E2	0.1155814 0.0943370	5.104888147-E4
0.5	0.1248396	0.1252707 0.07850367	3.453231186-E2	0.1248449 0.0899283	4.245447759-E4

(3-3) Electrical Circuit Engineering

Consider an electric circuit consisting of a resistance R, inductance L, a condenser of capacity C and electromotive power of voltage E in a series. A switch is also connected in the circuit, as shown in figure (2). Then by Kirchhoff's law, we have:



$L\frac{dI}{dt} + RI + Q\,C =$

Figure (2) An electric circuit

An electric circuit ,an inductance of 3 $henry$, a resistor of 4 $ohms$ and a capacitor of 5 $farad$ are connected in series with an emf of 8 sint $volts$. At t = 0, the charge on the capacitor and current in the circuit is zero. To obtain the charge and current at any time of $t > 0$.
Assume Q and I be instantaneous charge and current respectively at time t. Then by Kirchhoff's law: $L\frac{dQ}{dt} + RI + Q\,C = E$,

$$\frac{d^2Q}{dt^2} + 4\frac{dQ}{dt} + 5Q = 8\,\text{sint} \ldots (since\ I = \frac{dQ}{dt}),(I' = \frac{d^2Q}{dt^2}), \qquad \cdots (12)$$

with initial condition I(0)=Q(0)=0 ,

the exact solution of equation (12) is $(1+e^{-2t})sint - (1 - e^{-2t})cost$
After convarting (12) to systemof order one ,then can be solved by Runge-Kutta method as in table (3).

Table(3) The solution of electric circuit by second and fourth order of Runge- Kutta

t_n	Exacte S. y_{i+1} z_{i+1}	RK2 y_{i+1} z_{i+1}	Error (y_{i+1}) Error (z_{i+1})	RK4 y_{i+1} z_{i+1}	Error (y_{i+1}) Error (z_{i+1})
0.1	0.001206	0 0.03993336	1	0.001199 0.03500208	5.804311774-E3
0.2	0.0087330	0.0071880 0.1295841	0.1769151494	0.008722083 0.122286450	1.250085881-E3
0.3	0.0266684	0.0253218 0.24789111	0.05049421788	0.0266556 0.240032477	4.79968802-E4
0.4	0.0571936	0.0563408 0.37893214	0.01491075925	0.0571807 0.3717213090	2.255497119-E4
0.5	0.1010583	0.1008236 0.51089482	2.32242181-E2	0.1010465 0.505082305	1.167642836-E4

Form comparison Runge-Kutta methods in tables 1, 2, 3 with exact solution, we conclude the fourth order method is the most accurate for analytical solution.

REFRENCES:

[1] A.Kilicman and H.E.Gadain. "An application of double Laplac transform and sumudu transform', Lobachevskii J. Math.30 (3) (2009),

[2] AN ALbukhuttar and WH ALshemary. “Novel Transformations for Solving Ordinary Differential Equations with Variable Coefficients”, International Journal of Academic Multidisciplinary Research(IJAMR), Vol. 4(6), 2020

[3] Elzaki TM. On the connections between Laplace and Elzaki transforms. Advances in Theoretical and Applied Mathematics. 2011; 6(1).

[4] Endre SMayer D. F., An introduction to Numerical Analysis, Cambridge University Press, 2003

[5] F. Abdulmohdi, H. N. Kadhim and A. N .ALbukhuttar , 2020, "Shehu transform forActivityCosting System", Elixir International Journal, Vol.(144).

[6] F Naeimi Dafchahi, "A new refinement of jacobi method for solution of linear system equations Ax= b"Int. J. Contemp. Math. Sciences 3 (17), 2008 .

[7] J. D. Lambert, Numerical Methods for Ordinary Differential Systems, The Initial Value Problem, John Wiley & Sons, London, UK, 1991.

[8] J. H. Verner, Some Runge-Kutta formula pairs, SIAM J. Numer. Anal., 28 (1991), pp. 496

[9] Kasim Hussain, Fudziah Ismail, and Norazak Senu, Runge-Kutta Type Methods for Directly Solving Special Fourth-Order Ordinary Differential Equations, Hindawi Publishing Corporation Mathematical Problems in Engineering Volume 2015, Article ID 893763.

[10] Mehmet Sezer, AyşegülAkyüz-Daşcıog˘lu , A Taylor method for numerical solution of generalized pantograph equations with linear functional argument , Journal of Computational and Applied Mathematics 200 (1) , 2007.

[11] Masataka Usui, Hiroshi Niki, Toshiyuki Kohno,"Adaptive Gauss-Seidel method for linear systems", International Journal of Computer Mathematics 51 (1-2), 1994.

[12] Jain ,Iyengar and Saldanha,“Numerical solution of a fourth-order ordinary differential equation,” Jour nal of Engineering Mathematics, vol. 11, no. 4,1977.

[13] Tracogna S., Jackiewicz, A General Class of Two Step Runge-Kutta Method for ODEs, Springer Series in Computational Mathematics, Verlang 2010, 10(4).

[14]Taiwo O. A., Jimoh A. K., Bello A. K., Comparison of some Numerical Methods for the Solution of First and Second Orders Linear Integro Differential Equations, American Journal Of Engineering Research (AJER), 2014, 3(1).