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STUDYING THE ESTIMATION OF $\mathcal{N}(x)$ AND $\zeta(s)$ BY ADOPTING BALANZARIO'S METHOD

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***Abstract*:** According to Beurling in 1937, a generalized prime is any increasing infinite positive real sequence for which the first element is precisely greater than 1. Additionally, the basic fundamental theorem of arithmetic can be used to construct the sequence of Beurling integers. The example of a discrete and continuing Beurling's prime system is the main topic of this article. The challenge in this approach is finding a discrete system from a continues prime system. This work demonstrates the connection between Beurling Zeta function and Beurling counting functions which discusses the error -term of $\mathcal{N}(x)$ and $\pi(x)$ and the relation between them with an estimation in order to obtain a better error-term n comparison with what are known till now.

***Keywords*:** *(Analytic Number Theory, generalized prime systems)*

INTRODUCTION

Introduction

Beurling generalized numbers Beurling defined a set p of any increasing real value sequence of elements which be precisely greater than one.so, this indicate that $\{p_i\}_{i=1}^{\infty}$ called a generalized prime.

One can use the fundamental theorem of Arithmetic to generat the sequence $\{N_i\}_{i=1}^{\infty}$with n_0 to be the generalized integers or (Beurling integers)

The generalized counting functions of primes and of integers are defined inthe natural way as follows

$$\pi(x) = \sum_{p \le x} 1 \quad \& \quad \mathcal{N}(x) = \sum_{n \le x} 1$$

Suppose that $\mathcal{N}(x)$ satisfies for some δ and positive k, $|\mathcal{N}(x) - k| \le M\left(\frac{x}{log^{\delta}x}\right)$

Where Beurling had shown that if $\delta > 3/2$, then $\pi(x) \sim \frac{x}{logx}$ and moreover that is Diamond had shown that if $\delta \le {}^3/_2$ the condition with given$\delta$is so sharp.

The above assumption not exist that means the Beurling s condition is very sharp

The weird numbers let n be a natural number in N, where n is set to be abundant number if the sum of the proper divisor of n except itself is greater than n .

Additionally, n is deficient number if the sum of the proper divisors of n except itself is less than n .as well as n is set to be semiperfect if it is abundant and can expressed as a sum of some or all its distinct proper divisors

Now, from all of above explained of any natural number n we can defined the weird numbers to be any abundant number but not semi perfect.

So, many authors had been working in this field in order to give a general form of any weird number for example

Pajunen Showed in 1980 that any weird number is of the form $2^k p.q$ Where k integer number and p, q are

Moreover, G., Hasler, M.,Melfi, G.,Parton, they showed in 2019 that any weird number is of form

$x = c\prod_{i=1}^{r} p_r$ with c is deficient number and $p_1, p_2 \dots p_r$ are primes.

Balanzario s method Balanzario proved in his papers if

$$\pi(x) = \int_1^x \frac{1-t^{-\rho}}{logt} k(\log t)dt$$

Then there are positive numbers a and b such that for $x \geq x_0$

$$a\frac{x}{\log x} \leq \pi(x) \leq b\frac{x}{\log x}$$

And he obtained the error-term of $\pi(x)$ is $O(x^{1-\rho})$.

Also, he proved under some conditions

$$\mathcal{N}(x) = c_1 x + c_0 + A_m x cos(m\log x)\left(\frac{1}{logx}\right)^{1-\alpha_m} + O\left(\frac{xloglogx}{log^{2-\alpha_m}x}\right)$$

When ρ is a positive constant and this constant will determine the position of some singular points of Zeta function such that $0 < \rho < \frac{1}{2}$.

And he discussed the procedure for obtain N from π works when π is continuous as well as when it is discrete distribution by the formula.

$N(x) = \int_{1^-}^{x} dN = \int_{1^-}^{x} e^{d\pi}$

Now, from Balanzario's method we can chose any value of parameter ρ from its range it may be (0.1), m=1 and substitution this value in the error -term .it will get

$$\pi(x) = \int_e^x \frac{k(log\ t)}{log\ t} + O(x^{0.9}) \text{ and}$$

$$\mathcal{N}(x) = c_1 x + Axcos(\log x)\left(\frac{1}{logx}\right)^{1-\alpha} + O\left(\frac{xloglogx}{log^{2-\alpha}x}\right) \qquad (1)$$

Upper bound for $\zeta(s)$

As the reader see that the function $\zeta(\sigma + it) = O(e^t), (t > 0, t \to \infty)$ for any σ closed to the value $1/2$ which is known and for $\gamma \in [0,1)$, $\zeta(s)$ has an analytic continuation except for a simple pole at $s = 1$

with residue ρ and for $\sigma > \gamma$. So, if $|\mathcal{N}(x) - ax| \ll Rxe^{-g(x)}$ with a,R>0 and for some positive, increasing function g tending to the infinity such that $g'(x) = o\left(\frac{1}{x}\right)$ then for some c > 0 ,

$$\zeta(\sigma + it) = O(t^c)$$

with $1 - \frac{g\left(\frac{e^t}{t}\right)}{t} \leq \sigma \leq 1 - \frac{\log t}{t}$ where t is sufficiently large. For more details the reader could see [10].

So, applying the above and (1) obtaining

$$N(x) = \rho x + O\left(xexp^{-log\left(\frac{log^{2-\alpha}x}{loglogx}\right)}\right). \qquad (2)$$

Further, write $g\left(\frac{e^x}{x}\right) \sim \log\left(\frac{x^{2-\alpha}}{logx}\right)$, so

$g\left(\frac{e^x}{x}\right) \leq (1+\epsilon)log\left(\frac{x^{2-\alpha}}{logx}\right), \forall\epsilon > 0, x \geq x_0(\epsilon)$ Implies that

$$\zeta(\sigma + it) = O(t^c)\ for \qquad \sigma \geq 1 - \frac{\left((1+\epsilon)log\left(\frac{t^{2-\alpha}}{logt}\right)\right)}{t}$$

Since $\frac{log\left(\frac{t^{2-\alpha}}{logt}\right)}{t} \sim \frac{(2-\alpha)logt}{t}$ for $\sigma \geq 1 - \frac{(1+\epsilon)logt^{2-\alpha}}{t}$.

However, the error-term of $\mathcal{N}(x)$ in Balanzario' s method by using the forms of weird numbers

$$\frac{xloglogx}{log^{2-\alpha}x} = \frac{(c\prod_{i=1}^{r}p_i)\ loglog(c\prod_{i=1}^{r}p_i)}{(logc\prod_{i=1}^{r}p_i)^{2-\alpha}}$$

$$= \frac{c\prod_{i=1}^{r}p_i)loglogc + lo\ \ \sum_{i=1}^{r}logp_i}{(logc + \sum_{i=1}^{r}logp_i)^{2-\alpha}}$$

$$= \frac{c\prod_{i=1}^{r}p_i)loglogc + log\sum_{i=1}^{r}logp_i}{(logc\ \ \sum_{i=1}^{r}logp_i)^{2-\alpha}}$$

Since $\sum_{i=1}^{r} log\ p_i \leq a\ logp_r$ then

$log\sum_{i=1}^{r}logp_i \leq a\ logp_r$, a is constant

And

$$\sum_{i=1}^{r} logp_i \geq (logp_r)^{\frac{1}{2}}$$

Then $(logc + \sum_{i=1}^{r}logp_i)^{2-\alpha} \geq (logp_r)^{1-\frac{\alpha}{2}}$

$$=(logp_r)^{\beta} \quad \forall\beta < 1$$

Since $c\prod_{i=1}^{r}p_i \leq b\ p_r$,such that $b > c$

Finally, it will be

$$\frac{(c\prod_{i=1}^{r}p_i)\ loglog(c\prod_{i=1}^{r}p_i)}{(logc\prod_{i=1}^{r}p_i)^{2-\alpha}} \sim \frac{b\ p_{r\,(loglog\ c + a\ logp_r)}}{(logp_r)^{\beta}}$$

Theorem 1

$$\zeta(s) = \frac{s-0.9}{s-1} \prod_{0<|j|<n} (1 - \frac{0.1}{s - \mathrm{i}j - 0.9})^{-\alpha_j}$$

For $m \in (0, n)$ set

$$Ic_m = \frac{1}{2\pi!} \int_{c_m} \zeta(s) \frac{x^s}{s} \mathrm{d}s$$

With C_m the contour in figure Then for every such m, there is a real number A_m distinct from zero such that

$$Ic_m + Ic_{-m} = A_m x \cos(m \log x) \left(\frac{1}{\log x}\right)^{1-\alpha_m} + O\left(\frac{xlogloglogx}{\text{loglog}^{\,k-1-\alpha_m} x}\right)$$

Theorem 2

$$\pi(x) = \int_1^x \frac{1-t^{-0.1}}{\log t} G(\log t) dt$$

With

$G(x) = 1 + 2\alpha \cos(x)$

Where α is a fixed real number such that $0 < |\alpha| < 1$ (Note that $G(x) > 0$)

Let

$$\mathcal{N}(x) = \int_{1-}^{x} e^{\mathrm{d}\pi}$$

Then

$$\mathcal{N}(x) = c_1 x + \acute{A_1} x \cos(\log x) \left(\frac{1}{\log x}\right)^{1-\alpha} + O\left(\frac{xlogloglogx}{\text{loglog}^{\,k-1-\alpha} x}\right) \qquad (4)$$

From above and that the best estimate for k one can take it to be $2 < k < 2.5$

If we applied the last theorem on the error-term of $\mathcal{N}(x)$) from above proposition it will get

From a above theorem and (3) we can write

$$\mathcal{N}(x) = \rho x + O\left(xexp^{-log\left(\frac{loglog^{k-1-\alpha}x}{logloglogx}\right)}\right) \qquad (5)$$

Then $g(x) = log\left(\frac{loglog^{k-1-\alpha}x}{logloglogx}\right)$

Further we write $g\left(\frac{e^x}{x}\right) \sim \log\left(\frac{log^{k-1-\alpha}x}{loglogx}\right)$

Such that

$g\left(\frac{e^x}{x}\right) \leq (1+\epsilon) log\left(\frac{log^{k-1-\alpha}x}{loglogx}\right), \forall \epsilon > 0\,, x \geq x_0(\epsilon)$ Implies that

$$\zeta(\sigma + it) = O(t^c)\ for \qquad \sigma \geq 1 - \frac{(1+\epsilon)log\left(\frac{loglog^{k-1-\alpha}t}{loglogt}\right)}{t}$$

Since $\frac{log\left(\frac{loglog^{k-1-\alpha}t}{loglogt}\right)}{t} \sim \frac{loglog^{k-1-\alpha}t}{t}$ Then $\sigma \geq 1 - \frac{loglog^{k-1-\alpha}t}{t}$

Also, We can write the error-term of $\mathcal{N}(x)$ by using the one of the forms of weird numbers

$$= \frac{(c\prod_{i=1}^{r} p_i)\, logloglog(c\prod_{i=1}^{r} p_i)}{(loglogc\prod_{i=1}^{r} p_i)^{k-1-\alpha}}$$

$$= \frac{c\prod_{i=1}^{r} p_i) logloglogc + loglog\sum_{i=1}^{r} logp_i}{(loglogc + log\sum_{i=1}^{r} logp_i)^{2-\alpha}}$$

Since $\sum_{i=1}^{r} logp_i \leq a\, logp_r$

$loglog\sum_{i=1}^{r} logp_i \leq loga + log\, logp_r$ then

*A*nd

$$log\sum_{i=1}^{r} logp_i \geq (loglogp_r)^{\frac{1}{2}}$$

Then $(loglogc + log\sum_{i=1}^{r} logp_i)^{k-1-\alpha} \geq (loglogp_r)^{\frac{k-1-\alpha}{2}}$

$$=(loglogp_r)^{\beta} \quad \forall\beta < 1$$

Finally, it will be

$$\frac{(c\prod_{i=1}^{r} p_i)\, logloglog(c\prod_{i=1}^{r} p_i)}{(logc\prod_{i=1}^{r} p_i)^{k-1-\alpha}}$$

$$\sim \frac{(b.p_r)(logloglogc + logloga + loglogp_r)}{(loglogp_r)^{\beta}}$$

DISCUSSION AND FUTURE WORKS

The aim of this paper is to introduce the estimation of $\mathcal{N}(x)$ as x goes to infinity and $\zeta(s)$ for the real part lies in (0,1). So, the results noticed that this change of $|\mathcal{N}(x) - ax|$ for some positive constant a, was related to the changing of the value of σ (the real part of s)and hence its effects on the function $\zeta(s)$ as known. So, one may ask a crucial question "What is the best possible value of σ?" or one could look at to the above results and compare it when the Riemann-Hypothesis hold.

term. It's obvious that the attention of the authors was concentrated on estimation of O-result for $\mathcal{N}(x)$and $\zeta(s)$, However one can do more effort by using this estimation to camper, it with other result firsts and pick up an $\Omega -$ result for $\mathcal{N}(x)$. This opens a new window for the research one can do it in the future work.

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