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THE SOLUTION OF APPLIED LINEAR SYSTEMS USING ELZAKI TRANSFORMATION

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Abstract: In this paper, general formulas of solutions are derived to solve linear systems of first order or single equation of second order which is converted to linear system of first order. Moreover, these solution formulas have been employed to solve initial value problems that have applications in other sciences such as tank systems, pendulum systems, and electrical circuits.
Keyword: Linear systems of first order, Elzaki transformation, Tank system.

1. Introduction

Differential equation have been used to some degree in every branch of applied mathematics, Physics and engineering [3]. The employment of various integral transform to solve differential equations received a lot of scholarly attention[5] which is frequently used to solve ordinary and partial differential equations [1].Moreover, they are used for solving many initial value problems that difficult to solve with traditional methods, such as the Laplace, Temem, Shabhan transformations[2،4، 6].

One of the efficient integral transformation that is the focus of our study in this work is Elzaki transformation [10]. we consider function in the set μ defined by the Elzaki transformation:

$$\mu = \left\{\eta(x): \exists M_1\varphi_1, \varphi_2 > 0, |\eta(x)| < e^{\frac{|x|}{\varphi_j}}\, if\ x \in (-1)^j \times [0, \infty)\right\}.$$

where the function of exponential order in the set μ defined by

$$E[\eta(x)] = T(w) = w\int_0^\infty \eta(x)e^{-\frac{x}{w}}\,dx, \quad x \geq 0, \quad \varphi_1 \leq w < \varphi_2.$$

Elzaki is a useful technique for solving linear differential equations either single or systems with constants or variable coefficients in the time domain [8،9] while others used it in solve fuzzy ordinary differential equation[7].

2-Properties and theorems

In this section, some definitions and fundamental requirement in the following work are introduced.

2.1 Property: (linear property)

The Elzaki transform is characterized by the linear property, that is

$$E[c_1Q(t) \pm \cdots \pm c_nQ_n(t)] = c_1E\big(Q_1(t)\big) \pm \cdots \pm c_nE\big(Q_n(t)\big),$$

where $c_1, \ldots, c_n$ are constants, the functions $Q_1(t), \ldots, Q_n(t)$ are defined in $t \in (0, \infty)$.

2.2 Theorem : Let $T(w)$ is the Elzaki transformation of $\big[[E(\eta(x))] = T(w)\big]$.

i. $aki\ transform\ is\ characterized\ by\ the\ linear\ property, that\ is\ r\ function\ of\ exponential\ o$ $\frac{T(w)}{w} - w\eta(0)$

ii. $E[\eta``(x)] = \frac{T(w)}{w^2} - w\eta`(0) - \eta(0)$

iii. $E\left[\eta^{(n)}(x)\right] = \frac{T(w)}{w^n} - \sum_{k=0}^{n-1} \quad w^{2-n+k}\eta^k(0)$

where

$E[\eta`(x)] = w\int_0^\infty \quad \eta`(x)e^{-\frac{x}{w}}\,dx = \frac{T(w)}{w} - w\,\eta(0)$.

2.3 General formula for system of first order in dimension n.

Consider the non-homogenous system of first order

$(\eta_1(t) \ \vdots \ \eta_n(t)\ )' = (\varphi_{11} \ \vdots \ \cdots \quad \varphi_{1n} \ \vdots \ \varphi_{n1} \ \cdots \ \varphi_{nn}\ )(\eta_1(t) \ \vdots \ \eta_n(t)\ ) + (\beta_1(t) \ \vdots \ \beta_n(t)\ )$ (2-1)

where $\varphi_{ij}: i, j = 1, \ldots, n$ are constants coefficient and $\eta_1(t), \ldots, \eta_n(t), \beta_1(t), \ldots, \beta_n(t)$ are functions of t.

2.4 Reducing n-order linear equation to system of first order.

Let $\psi_t^{(n)} + a_1\psi_{(t)}^{(n-1)} + \cdots + a_{n-1}\psi_{(t)} = f(t)$ (2-2)

$a_1, \ldots, a_{n-1}$ are constants. $\psi_{(t)}$ and $f(t)$ are functions of t.

To convert equation (2-2) to system (2-1), we assumption

$\eta_1(t) = \psi(t)$

$\eta_2(t) = \psi'_{(t)} = \eta'_1(t)$

$\eta_3(t) = \psi''_{(t)} = \eta''_2(t)$

$\vdots \qquad \vdots$

$\eta_n(t) = \psi_{(t)}^{(n-1)} = \eta'_{n-1}(t)$

$\eta'_n(t) = \psi_{(t)}^{n}$.

Therefore, equation (2-2) become

$(\eta_1(t) \ \vdots\vdots\vdots \ \eta_n(t)\ )' = (0\ 1\ 0 \quad 0 \quad \cdots \quad 0 \ \ 0\ 0\ 1 \quad 0 \quad \cdots \quad 0 \ \ 0 \ \vdots \ a_{n-1} \ \ 0 \ \vdots \ a_{n-2} \ \ 0 \quad 1 \quad \cdots \quad 0 \ \vdots \quad \vdots \qquad \vdots \ a_{n-3} \ \ a_{n-4} \quad \cdots \quad a_1 \quad) +$

$(0\ 0\ 0 \ \ \vdots \ f(t)\ )$ (2-3)

2.5 Elzaki transform for fundamental functions.

The following table (1-1) gives an Elzaki transformation of certain fundamental functions.

ID	function $N(x)$	$E\left(\eta(x)\right) = w\int_0^\infty \quad \eta(x) = e^{-\frac{x}{w}}dx$
1	*1*	w^2
2	x	w^3
3	x^n	$n!\,w^{n+2}$
4	e^{ax}	$\frac{w^2}{1-aw}$
5	$sin\ sin\ (ax)$	$\frac{aw^3}{1+a^2w^2}$
6	$cos\ cos\ ax$	$\frac{w^2}{1+a^2w^2}$
7	$x\ sin\ sin\ (ax)$	$\frac{2a^2w^4}{1+a^2w^2}$
8	$x\ cos\ cos\ (ax)$	$\frac{w(1-a^2w^2)}{(1+a^2w^2)^2}$

9	$sinh\, sinh\, (ax)$	$\frac{aw^3}{1-a^2w^2}$
10	$cosh\, cosh\, (ax)$	$\frac{w^2}{1-a^2w^2}$
11	e^{ax} $sin\, sin\, (bx)$	$\frac{bw^3}{(1+aw)^2+b^2w^2}$
12	e^{ax} $cos\, cos\, (bx)$	$\frac{(1-aw)w^2}{(1+aw)^2+b^2w^2}$

Table(1): Elzaki transform for fundamental functions .

3. The general Formula of solution to the non-homogenous system of first order in n-dimension.

Taking Elzaki transformation for system (2-1), yields:

$$\frac{E(\eta_1(t))}{w} - w\eta_1(0) = \varphi_{11}E(\eta_1(t)) + \varphi_{12}E(\eta_2(t)) + \cdots + \varphi_{1n}E(\eta_n(t)) + E(\beta_1(t))$$

$$\vdots \qquad\qquad \vdots$$

$$\frac{E(\eta_n(t))}{w} - w\eta_n(0) = \varphi_{n1}E(\eta_1(t)) + \varphi_{n2}E(\eta_2(t)) + \cdots + \varphi_{nn}E(\eta_n(t)) + E(\beta_n(t))$$

Where $\eta_1(0), \dots, \eta_n(0)$ are initial conditions.

$$\left(\frac{1}{w} - \varphi_{11}\right)E(\eta_1(t)) - \varphi_{12}E(\eta_2(t)) - \cdots - \varphi_{1n}E(\eta_n(t)) = w\eta_1(0) + E(\beta_1(t))$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\left(\frac{1}{w} - \varphi_{n1}\right)E(\eta_n(t)) - \varphi_{n2}E(\eta_2(t)) - \cdots - \varphi_{nn}E(\eta_n(t)) = w\eta_n(0) + E(\beta_n(t))$$

using Gramer's rule, such as:

$$\Delta = \left|\left(\frac{1}{w} - \varphi_{11}\right) - \varphi_{12} \cdots \varphi_{1n} - \varphi_{21} \left(\frac{1}{w} - \varphi_{22}\right) \cdots \varphi_{2n} \vdots -\varphi_{n1} \vdots -\varphi_{n2} \vdots \cdots \left(\frac{1}{w} - \varphi_{nn}\right)\right|$$

Also,

$$E(\eta_1(t)) = \frac{1}{\Delta}\left|w\eta_1(0) + E(\beta_1(t)) - \varphi_{12} - \varphi_{1n} \vdots\vdots\vdots\ w\eta_n(0) + E(\beta_n(t)) - \varphi_{n2} \left(\frac{1}{w} - \varphi_{nn}\right)\right| \qquad (3\text{-}1)$$

$$\vdots$$

$$E(\eta_n(t)) = \frac{1}{\Delta}\left|\left(\frac{1}{w} - \varphi_{11}\right) - \varphi_{12}\, w(\eta_1(0) + E(\beta_1(t)) - \varphi_{21} \vdots \left(\frac{1}{w} - \varphi_{22}\right) \vdots \cdots - \varphi_{n1} - \varphi_{n2} \cdots w\eta_n(0) + E(\beta_1(t))\right| \qquad (3\text{-}2)$$

After taking inverse of Elzaki transform to $E(\eta_1(t)), \dots, E(\eta_n(t))$, can be obtion the solution of system (2-1).

4. The general formula of solution to the homogenous system of first order in dimension n.

If $\beta_j(t) = 0, j = 1, \dots, n$ in system (2-1), then it is called homogenous system. In similar steps of section (3), can obtained the solution as:

$$E(\eta_1(t)) = \frac{1}{\Delta}\left| w\eta_1(0) \ -\varphi_{12} \ \dots - \varphi_{1n} \ w\eta_2(0) \ \left(\frac{1}{w} - \varphi_{22}\right) \cdots - \varphi_{2n} \ \vdots \ w\eta_n(0) \ \vdots \ -\varphi_{n2} \ \vdots \ \cdots \left(\frac{1}{w} - \varphi_{nn}\right) \right| \quad (4\text{-}1)$$

$\vdots$

$$E(\eta_n(t)) = \frac{1}{\Delta}\left| \left(\frac{1}{w} - \varphi_{11}\right) \ -\varphi_{12} \ \cdots w\eta_1(0) \ -\varphi_{21} \ \left(\frac{1}{w} - \varphi_{22}\right) \cdots \vdots \vdots \ -\varphi_{n1} \ \vdots \ -\varphi_{n2} \ \cdots w\eta_n(0) \right| \quad (4\text{-}2)$$

In similar way taking inverse of Elzaki transform to (4-1) … (4-2) can be obtain the solution of homogenous system.

5. Application

In this section, we introduced supported examples that shows efficiency of derivate formulas. Such as Tank, Pendulum and Electric:

5-1 Tank system of equation:

One of the important application of mathematical system is tank, which has the general formula

$$\grave{\eta}_1(t) = \varphi_{11}\eta_1(t) + \varphi_{12}\eta_2(t) \ \grave{\eta}_2(t) = \varphi_{21}\eta_1(t) + \varphi_{22}\eta_2(t) \ ] \quad (5\text{-}1)$$

$\eta_1(0) = a_1 \ , \ \eta_2(0) = a_2$

This system represent tank connection activity with water inlet and exist resources.

$$E(\eta_1(t)) = w^2\left[\frac{a_1 - (a_1\varphi_4 - a_2\varphi_2)w}{(\varphi_1\varphi_4 - \varphi_2\varphi_3)w^2 - (\varphi_1 + \varphi_4)w + 1}\right]$$

$$E(\eta_2(t)) = w^2\left[\frac{a_2 - (a_2\varphi_1 - a_1\varphi_3)w}{(\varphi_1\varphi_4 - \varphi_2\varphi_3)w^2 - (\varphi_1 + \varphi\ )w + 1}\right]$$

Taking E^{-1} for above equations :

$$\eta_1(t) = E^{-1}\left[w^2\left[\frac{a_1 - (a_1\varphi_4 - a_2\varphi_2)w}{(\varphi_1\varphi_4 - \varphi_2\varphi_3)w^2 - (\varphi_1 + \varphi_4)w + 1}\right]\right] \quad (5\text{-}2)$$

$$\eta_2(t) = E^{-1}\left[w^2\left[\frac{a_2 - (a_2\varphi_1 - a_1\varphi_3)w}{(\varphi_1\varphi_4 - \varphi_2\varphi_3)w^2 - (\varphi_1 + \varphi_4)w + 1}\right]\right] \quad (5\text{-}3)$$

$\eta_1(t)$ and $\eta_2(t)$ repersant the solution of system (5-1)

For example, if has two tank as in figure (1) where each holding 24 liters of seawater, and a cylinder connecting them. Water is transported under pressure at a rate of 8L/m from the first tank T_1, to the second tank T_2. And with reduced pressure, water moved at a rate of 2L/m from Tank T_2 in the second Tank to Tank T_1 in the first. Moreover, fresh water is added to the first tank (T_1) at a speed of 6, and water is removed from the second tank at a similar speed (64m). As indicated in the illustration, salt is present in each of the two tanks. So, the following system is used to explain the problem if we want to determine the mass of salt present in each tank at a given moment.

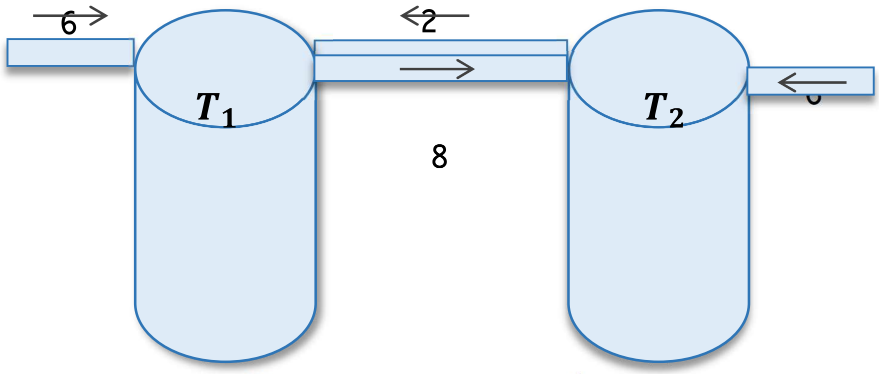

Figure (1): Tank Application

Consider problem as the system

$\grave{\eta}_1(t) = -\frac{1}{3}\eta_1 + \frac{1}{12}\eta_2$, $\eta_1(0) = 1$

$\grave{\eta}_2(t) = \frac{1}{3}\eta_1 - \frac{1}{3}\eta_2$, $\eta_2(0) = 6$

By using formula (5-2) and (5-3). We get

$$\eta_{1(t)} = E^{-1}\left[w^2\left[\frac{a_1-(a_1\varphi_4-a_2\varphi_2)w}{(\varphi_2\varphi_4-\varphi_2\varphi_3)w^2-(\varphi_1+\varphi_4)w+1}\right]\right]$$

$$\eta_2(t) = E^{-1}\left[w^2\left[\frac{a_2-(a_2\varphi_1-a_1\varphi_3)w}{(\varphi_2\varphi_4-\varphi_2\varphi_3)w^2-(\varphi_1+\varphi_4)w+1}\right]\right]$$

Simplification

$$\eta_1(t) = E^{-1}\left[w^2\left[\frac{1+\frac{5}{6}w}{\frac{1}{12}w^2+\frac{2}{3}w+1}\right]\right]$$

$$\eta_2(t) = E^{-1}\left[w^2\left[\frac{6+\frac{7}{3}w}{\frac{1}{12}w^2+\frac{2}{3}w+1}\right]\right]$$

$$\eta_1(t) = -e^{-\frac{1}{2}t} + 2e^{-\frac{1}{6}t}$$

$$\eta_2(t) = 2e^{-\frac{1}{2}t} + 4e^{-\frac{1}{6}t}$$

where $\eta_1(t)$ and $\eta_2(t)$ represent salt mass that collects in the tanks over time.

5.2 Pendulum Application

Pendulum is another application of mathematical system, which has the general formula.

$$\eta``(t) + \chi\eta`(t) + \delta\eta(t) = \beta(t) \qquad (5\text{-}2)$$

where χ and δ are constants, with $\eta(0) = a_1, \eta`(0) = a_2$

using formula (2-3) to convert equation (5-2) to system of first order.

$\grave{\eta}_1(t) = \eta_2(t)$, $\qquad \eta_1(0) = a_1$

$\grave{\eta}_2(t) = -\chi\eta_1(t) - \delta\eta_2(t) + \beta(t) \qquad \eta_2(0) = a_2$

From formula (3.1) – (3.2) to obtain the solution

$$E[\eta_1(t)] = w^2\left[\frac{a_1+(a_1\chi+a_2)w}{1+\chi w+\delta\ \ ^2}\right]$$

$$E[\eta_2(t)] = w^2\left[\frac{a_2-a_1\delta w}{1+\chi w+\delta\ \ ^2}\right]$$

Taking inverse of Elzaki after simple calculation

$$\eta_1(t) = E^{-1}\left[w^2\left[\frac{a_1+(a_1\chi+a_2)w}{1+\chi w+\delta w^2}\right]\right] \quad (5\text{-}4)$$

$$\eta_2(t) = E^{-1}\left[w^2\left[\frac{a_2-a_1\delta w}{1+\chi w+\delta w^2}\right]\right] \quad (5\text{-}5)$$

$\eta_1(t)$ and $\eta_2(t)$ represent the solution of system (5-2)
For example if we take $\chi = 5$ and $\delta = 4$, $a_1 = 0, a_2 = 1, \beta(t) = e^t$
Then equation (5-2) become $\eta``(t) + 5\eta`(t) + 4\eta(t) = e^t$
By using (5.4) – (5.5) to find the particular solution of above equation.

$$\eta_1(t) = E^{-1}\left[w^2\left[\frac{1+\left(5+\frac{w}{1-w}\right)w}{1+5w+4w^2}\right]\right]$$

$$\eta_2(t) = E^{-1}\left[w^2\left[\frac{-3+4w^2}{1+5w+4w^2}\right]\right]$$

$$\eta_1(t) = \frac{1}{10}e^t + \frac{7}{6}e^{-t} - \frac{4}{15}e^{-4t}$$

$$\eta_2(t) = \frac{1}{10}e^t - \frac{7}{6}e^{-t} + \frac{16}{15}e^{-4t}$$

$\eta_1(t)$ and $\eta_2(t)$ represent the solution of converting system.

5.3 Electric Application

An electric circuit can be expressed as a system of first order, which has the form

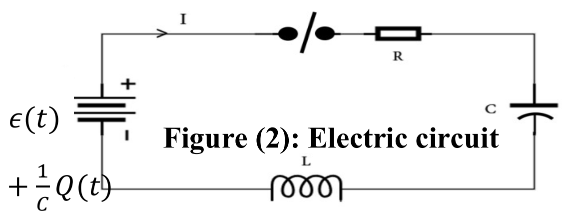

Figure (2): Electric circuit

$$\epsilon = RI(t) + LI'(t) + \frac{1}{C}Q(t)$$

R; ohms = resistance
L; henrys = inductance
C; farads = capacitance
Since $I(t) = Q'(t)$ and using formula (2.3) yield :

$$Q'(t) = S(t)$$

$$S'(t) = -\frac{R}{L}S(t) - \frac{1}{CL}Q(t) + \frac{\epsilon}{L}$$

The solution of above system can be obtained by using the general formula of (3.1) - (3.2)

$$Q(t) = E^{-1}\big[wE[S]\big] \quad (5\text{-}6)$$

$$S(t) = E^{-1}\left[\frac{wE[\epsilon]}{L+Rw+\frac{1}{C}w^2}\right] \quad (5\text{-}7)$$

For example: An electric circuit with an emf of 10 volts, 1 Henry inductance, 4 ohm resistor, and 5 Henry capacitor are farad linked in series. The capacitor's charge and the circuit's current are both zero at time t. At any moment after $t > 0$, to switch the charge and current.
Suppose that at time t, Q and I represent the respective instantaneous charge and current.
Then by kirchhoff's law:

$$Q''(t) + 4Q'(t) + 5Q(t) = 10$$

sine $I = Q'$, $I' = Q''$

$$Q' = S$$

$S' = -4S - 5Q + 10$

Using formula (5-6) , (5-7) to find charge in capacitor

$S(t) = 10e^{-2t} \sin \sin t$

$Q(t) = 2(1 - e^{-2t} \cos \cos t\ - 2e^{-2t} \sin \sin t\ )$

where $S(t)$ and $Q(t)$ represent current and charge respectively.

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