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A NUMERICAL STUDY OF QUASI LINEAR DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER USING THE METHOD OF COLLOCATION

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Abstract
The quasi-linear fractional-order ordinary differential equations(FODEs) have been examined in this study. The collocation approach takes into account a numerical solution of this kind. For solving FODEs, the B-spline basis has been taken into consideration. To compare the numerical and analytical answers for the test examples, several of the implementations examples are used. They have shown that the recently developed approach's numerical results and analytical solutions are in good agreement, showing the effectiveness and accuracy of the collection method and B- cubic spline combination.
Keyword: Quasi-linear, Fractional order, Ordinary Differential Equations; Order; DEs; ODEs, FDEs

Introduction

One of the most significant subfields of applied mathematics is differential equation (DE), which finds use in both science and engineering. Numerous forms of DEs are used in the majority of mathematical modeling in science and engineering. Additionally, DEs play a significant role in a number of applications of mathematics, including those in physics, engineering, biology, medicine, chemistry, and economics. The tools of differential equations (DEs), in particular frac- tional differential equations (FDEs), are used to describe the mathematical representations of the real situations in applied science and engineering. Numerous traditional or contemporary analyt- ical and numerical techniques have been investigated for a very long time to solve DEs Farlow (2012). Significant researches on fractional calculus(FC) was published in applied scientific and engineering fields in the 20th century. Numerous applications in the various fields of biologi- cal models, mathematical models of fluid mechanics and electrochemistry have been described by using the advancements in fractional calculus. For many types of DEs, either analytically or numerically, it had been challenging for mathematicians to apply their imagination and find the answers. The researchers of engineers or Scientists can currently use several kinds of powerful classical and modern numerical and analytical approaches. Following is a collection of the lit- erature review that covers numerous modern approaches for solving mathematical models that contain DEs: Bhrawy et al. (2014) introduced a new Jacobi spectral collocation method(CM) for solving 1+1 Schrödinger FDEs and fractional-coupled Schrödinger systems, while Awoyemi

(2005) introduced an algorithmic CM for direct solutions of IVPs of fourth-order. The fractional Fokker-Planck equations were solved by Hafez et al. (2015) using the Jacobi Gauss-Lobatto and Gauss-Radau collocation algorithms, and the fractional optimum control issues have been solved by Yousefi et al. (2011) using the Legendre multiwavelet CM. For literature review of the spline researches, spline regression is a method that Marsh & Cormier (2001) proposed for smoothing out and fitting timeline kinks while Zhang et al. (2019) introduced a generalized cubic exponential B-spline technique that can produce a variety of curves. The cubic spline interpolation pooling approach, which Huang et al. (2020) proposed and is suitable for processing one-dimensional signals, was described. By utilizing the excellent fitting effects of the cubic spline interpolation method that Balestriero et al. (2018) used to construct the fitting function, the proposed method can convert the pooling problem into a linear fitting problem. Lastly, spline functions and oper- ators were used by Balestriero et al. (2018) to firmly connect deep networks (DNs) and approxi- mation theory.

Finally, Tirmizi et al. (2008) developed a non-polynomial splines method for solving 6th-order BVPs. Not all DEs may be directly or indirectly solved using analytical methods, whether they be solved directly or indirectly. We are forced to examine the suggested numerical methods by this proposition. Numerous researchers, including Mechee & Senu (2012), presented a numer- ical study for fractional Lane-Emden differential equations using the collocation method. For nonlinear FDEs, while Chen et al. (2012) looked at the error analysis for the numerical solution of FDEs using the Haar wavelets(HWs) approach, Saeed & ur Rehman (2013) used the (HWs) quasi-linearization methodology and, Li & Hu (2010) and Saeedi et al. (2011) investigated and applied the operational HWs method for fractional Volterra integral equations.

The quasi-linear fractional-order ODEs have been investigated in this study. This type of DE is solved using the CM with a fractional B-cubic spline basis. The proposed method's exact solutions have been used to compare the solutions for the test examples. This comparison demonstrated the created method's effectiveness and precision.

Preliminary

We have provided some background details and RK technique history in this section which is related to the study's challenges.

Fractional Calculus

Applications of fractional calculus(FCs) have significance in a variety fields of engineering, sci- entific, and mathematical modeling. The class of FDEs of various types plays important roles and tools in both of physics, and mathematics. Many studies on FCs and FDEs, involving different operators such as Caputo, and Riemann-Liouville operators have appeared during the past three decades. Applications of fractional calculus (FCs) have significance in many areas of mathemat- ical, scientific, and engineering modeling. The benefits of fractional derivatives can be demon- strated by modeling the mechanical and electrical properties of actual materials as well as by explaining the properties of gases, liquids, and rocks. Various types of FDEs play significant roles and serve as useful tools in both physics and mathematics. Over the past three decades, a number of studies on FCs and FDEs utilizing different operators, including Caputo and Riemann-Liouville operators, have been published.

Types of Quasi-Linear FDEs of nth-Order

The following form is the general class of quasi-linear n^{th}-order FDEs:

$$D^{n\alpha} w(\varsigma) = \Phi(\varsigma, w(\varsigma), w'(\varsigma), w''(\varsigma), w'''(\varsigma), \dots, w^{(n)}(\varsigma)); \qquad a < \varsigma < b, 0 < \alpha < 1, \qquad (1)$$

From the general class of quasi linear FDEs of n^{th}-order in the Equation (1), we have the specialclass of quasi-linear, third-order FDEs in case n=3.

A Class of Quasi-Linear FDEs of Third-Order

$$D^{3\alpha} w(\varsigma) = \Phi(\varsigma, w(\varsigma), w'(\varsigma), w''(\varsigma), w'''(\varsigma)); \qquad 0 < \varsigma < b, 0 < \alpha < 1, \qquad (2)$$

On the other hand, the general class of quasi-linear nth-order FDE in Equation (1), when n=2, we have a particular class of quasi-linear, second-order FDEs.

A Class of Quasi-Linear Second-Order FDEs

Consider the following quasi-linear second-order FDE:

$$D^{2\alpha} u(\varsigma) = \Phi(t, w(\varsigma), w'(\varsigma), w''(\varsigma)); \quad 0 < \varsigma < b, 0 < \alpha < 1, \qquad (3)$$

Also, from the general class of quasi linear nth-order fractional ordinary differential equation in the Equation (1), we have the special class of quasi-linear, first-order fractional ODEs in case n=1.

A Class of Quasi-Linear First-Order Fractional ODEs

A quasi-linear first-order FDE is as follows:

$$D^{\alpha} w(\varsigma) = \Phi(\varsigma, w(\varsigma), w'(\varsigma)); \quad 0 < \varsigma < b, 0 < \alpha < 1, \qquad (4)$$

Boundary Second-Order, Quasi-Linear Fractional Differential Equations(FRDEs)

The following is the form of quasi-linear FDE of second-order:

$$D^{2\alpha} w(\varsigma) = \Phi(\varsigma, w(\varsigma), w'(\varsigma), w''(\varsigma)); \qquad 0 < \varsigma < 1, 0 < \alpha < 1, \quad (5)$$

with the boundary conditions

$$w(0) = \xi_0; w(1) = \xi_1. (6)$$

Numerical Solutions of FDEs Using B-Cubic Spline

Spline Functions

Spline functions for interpolation are often chosen as minimizers of relevant roughness measures under the limitations of interpolation. The functions of smoothing splines are chosen to minimize a weighted combination of the average squared approximation error over observed data and the roughness measure, which can be thought of as generalizations of interpolation splines. The spline functions have been shown to be finite-dimensional in nature for a variety of significant definitions of the roughness measure, which is the primary explanation for their usefulness in computations and representation. The cubic B-spline basis uses in solving ODEs were addressed in this section. In situations in which data interpolation is necessary, the expression "spline" is used to signify a large class of smooth functions (Faires & Burden (2003)). The next part of this section only addresses one-dimensional polynomial splines and uses the term "spline" in a

particular manner. If the basis functions satisfy ϕi(ς) ∈ Cn−1(−∞, ∞) for i = 1, 2,... , n., ϕ(ς) = {ϕ1(ς), ϕ2(ς),..., ϕn(ς)} is referred to as a spline base of order n. Firstly, we partition the interval [0, 1] to n subintervals with the norm of partition h = n 1 . However, for each i = 0, 1,... , n + 1, we

have the equally-spaced nodes ςi = ih, and then,the spline basis functions {ϕ(ς)}n+1 are defined on the interval [0, 1]

B-Cubic Spline

The B-cubic spline basis, which is defined as follows, was employed by several researchers.

$$S(\varsigma) = \frac{1}{4}\begin{cases} 0 & ,\varsigma < -2 \\ (2+\varsigma)^3 & ,\ 2 \le \varsigma \le -1 \\ (2+\varsigma)^3 - 4(1+\varsigma)^3 & ,-1 < \varsigma \le 0 \\ (2+\varsigma)^3 - 4(1+\varsigma)^3 & ,0 < \varsigma \le 1 \\ (2+\varsigma)^3 & ,1 < \varsigma \le 2 \\ 0 & ,\ \varsigma > 2 \end{cases} \tag{7}$$

Consequently, $S(\varsigma) \in C^2(-\infty, \infty)$. To create a cubic spline basis that complies with the bound-ary requirements $\varphi_i(0) = \varphi_i(1)$ for $i = 1, 2, \dots, n$ Following are the cubic spline functions we havecreated as elements:

$$\varphi_i(\varsigma) = \begin{cases} S\left(\frac{\varsigma}{h}\right) - 4S\left(\frac{\varsigma+h}{h}\right) & ,i = 0 \\ S\left(\frac{\varsigma-h}{h}\right) - S\left(\frac{\varsigma+h}{h}\right) & ,i = 1 \\ S\left(\frac{\varsigma-ih}{h}\right) & ,2 \le i \le n \\ S\left(\frac{\varsigma-nh}{h}\right) - S\left(\frac{\varsigma(n+2)h}{h}\right) & ,i = n \\ S\left(\frac{\varsigma-(n+1)h}{h}\right) - 4S\left(\frac{\varsigma-(n+2)h}{h}\right) & ,i = n+1 \end{cases} \tag{8}$$

In the following Figure 1 represents the cubic spline functions.

Table 1: Values at node points Cubic B-Spline

ς_i	$\varphi_i(\varsigma_i)$	$\varphi_i'(\varsigma_i)$	$\varphi_i''(\varsigma_i)$
ς_{i-2}	0	0	0
ς_{i-1}	$\frac{1}{4}$	$\frac{3}{4}$	$-\frac{3}{2}$
ς_i	1	0	$-\frac{3}{4}$
ς_{i+1}	$\frac{1}{4}$	$-\frac{3}{4}$	$-\frac{3}{2}$
ς_{i+2}	0	0	0

B-Cubic Spline

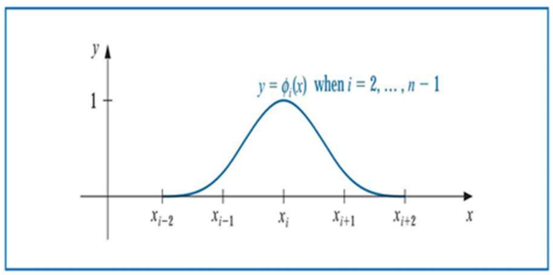
This subsection introduced the cubic B-spline operational matrix FSα of integration of the fractional order as follows:

The B-cubic spline operational FSal pha of integration of the fractional order was introduced in this subsection as follows:

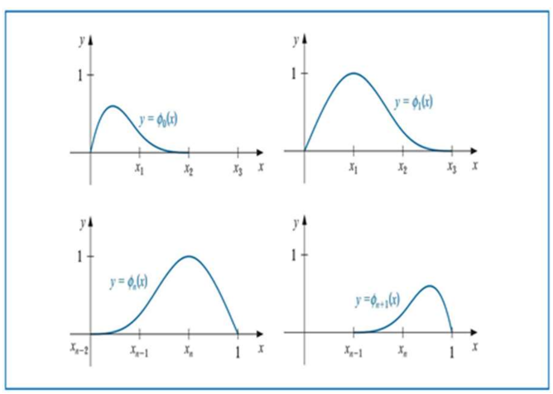
$$J_{\tau}^{\alpha}(\tau)=\frac{1}{\Gamma(\alpha+4)}\begin{cases}0 & ,\ \tau<-2\\ \frac{3}{2}\,\tau^{\alpha+3} & ,-2\le\tau\le-1\\ \frac{3}{2}\,\tau^{\alpha+3}-6\,\tau_1{}^{\alpha+3} & ,-1<\tau\le 0\\ \frac{3}{2}\,\tau^{\alpha+3}-6\,\tau_1{}^{\alpha+3}+9\,\tau_2{}^{\alpha+3} & ,0<\tau\le 1\\ \frac{3}{2}\,\tau^{\alpha+3}-6\,\tau_1{}^{\alpha+3}+9\,\tau_2{}^{\alpha+3}-6\,\tau_3{}^{\alpha+3} & ,1<\tau\le 2\\ 0 & ,\quad \tau>2\end{cases}$$

(9)

where $\tau_1=\tau-1$, $\tau_2=\tau-2$ and $\tau_3=\tau-3$



(1)



(2)

Figure 1: (2) B-Cubic Spline Function and (b) Compound B-Cubic Spline Function

The Collection Method Analysis

Establish the node points τi = a + ih for i = 0, 1,... , n . Discretizing the functions

$$\varphi(\tau)=\{\varphi_1(\tau),\varphi_2(\tau),\varphi_3(\tau),\dots,\varphi_n(\tau)\}.$$

Suppose

$$w(\tau) = \sum_{i=1} c_i \varphi_i(\tau). \quad (10)$$

Fix $\tau = \tau_j$ in the form $w(\tau)$ in the DE, we get the coefficient in matrix formula $\varphi_{i,j} = \varphi_i(\tau_j)$ and $\varphi^{J}_{i,j} = \varphi^{J}_{i}(\tau_j)$. The dimension of matrix coefficients has is nxn. For any differentiable function $w(\tau)$ in the domain $(0, 1)$ can be written as an finite sum of spline-base.

3.3.1 The Quasi-Linear FDEs of Second-Order

Take into account the general second-order FODE with the boundary conditions provided by Equations (5) and Equation (6).

By substituting the Equation (10) which satisfy the BCs in Equation (5) at fixd point $\tau = \tau_k$ for $k = 1, 2, \dots, n$. to get

$$\sum_{i=1}^{n} c_i D^{\alpha} (\varphi_i(\tau_j)) = \varphi(\tau_j, \sum_{i=1}^{n} c_i \varphi_i(\tau_j), \sum_{i=1}^{n} c_i \varphi'_i(\tau_j), \sum_{i=1}^{n} c_i \varphi''_i(\tau_j)); \quad 0 < \tau < 1, 0 < \alpha < 1, \quad (11)$$

However, we have a system of n algebraic equations, where the coefficients are the unknowns c_i for $i = 1, 2, \dots, n$. If the function φ is linear, then this system is a linear system. The coefficients matrix in this instance has the following formula:

$$B_{ij} = \Theta_i(\vec{t}, \vec{\Phi})$$

and

$$b_i = \Omega_i(\vec{t})$$

for j, $i = 1, 2, \dots, n$. Hence, We can find the approximation of the coefficients of the solution's by solving the linear system of the coefficients $Bx = c$.

4 Algorithm of The Proposed Method

The steps of the proposed approach can be summarized as follows to roughly approximate the solution of the boundary-value issue in Equations (5)-(6):

Step I: Firstly, Select an appropriate approximation base.

$$\Omega = \{\Omega_0(t), \Omega_1(t), \Omega_2(t), \dots, \Omega_m(t)\},$$

where $\Omega_j(0) = 0$ for $j = 1, 2, \dots, m$

Step II: Consider the solution of Equation (5) with the conditions in Equation (6) in the following form:

Consider the following form for the Equation (5)'s solution given given the conditions in (6):

$$W(\tau) = \sum_{i=1}^{N} c_i \Omega_i(\tau). \quad (12)$$

Step III: Input the endpoint b,the number of partition N, and the initial condition β_i for =0,1,2,. . . ,n-1.

Step IV: Put $h = \frac{b}{N}$ then, the interval $[a, b]$ has the nodes $t_1, t_2, t_3, \ldots, t_N$;

Step V: Put $t = t_j$ in Equation (12) to obtain

$$y(t_j) = \beta_0 + (\beta_1 - \beta_0)t_j + \sum_{i=1}^{N} a_i \Omega_i(t_j). \quad (13)$$

for $j = 1, 2, \ldots, N$

Step VI: Substitute the Equation (13) in the Equation (1) for j=1,2,. . . , N.

Step IX: The linear system $Ax = b$ is thus generated from the last equation and can be solved using any numerical approach for solving linear systems of algebraic equations.

5 Numerical Collocation Method

One of the most effective approximation techniques for solving FDEs is the collocation method. This method's foundation is the approximation of the FDE solution by a series of complete sequences of functions, where we define a complete sequence of functions as one that, generally speaking, has no nonzero function perpendicular to it. $w(\tau)$.

6 Implementation

In this section, the developed method examined in solving some examples of second-order FDEs using MATLAB software. A comparative simulation has been performed between the numerical results of the implementations and exact solutions in Figure 1.

Example 6.1. ***Linear***

$$D^{2\alpha} u(\varsigma) = u''(\varsigma) + 2u'(\varsigma) + u(\varsigma) + a^{2\alpha} e^{a\varsigma}, \qquad a \le \varsigma \le b, 0 < \alpha < 1$$

BCs: $u(0) = 0; \quad u(1) = 1;$

Exact solution: $u(\varsigma) = e^{a\varsigma}.$

Example 6.2. ***Nonlinear***

$$D^{2\alpha} u(\varsigma) = (\Gamma(2))^2 + (u''(\varsigma))^2 + u'(\varsigma) + 2t - 5, \qquad a \le \varsigma \le b, 0 < \alpha < 1$$

BCs: $u(0) = u(1) = 0;$

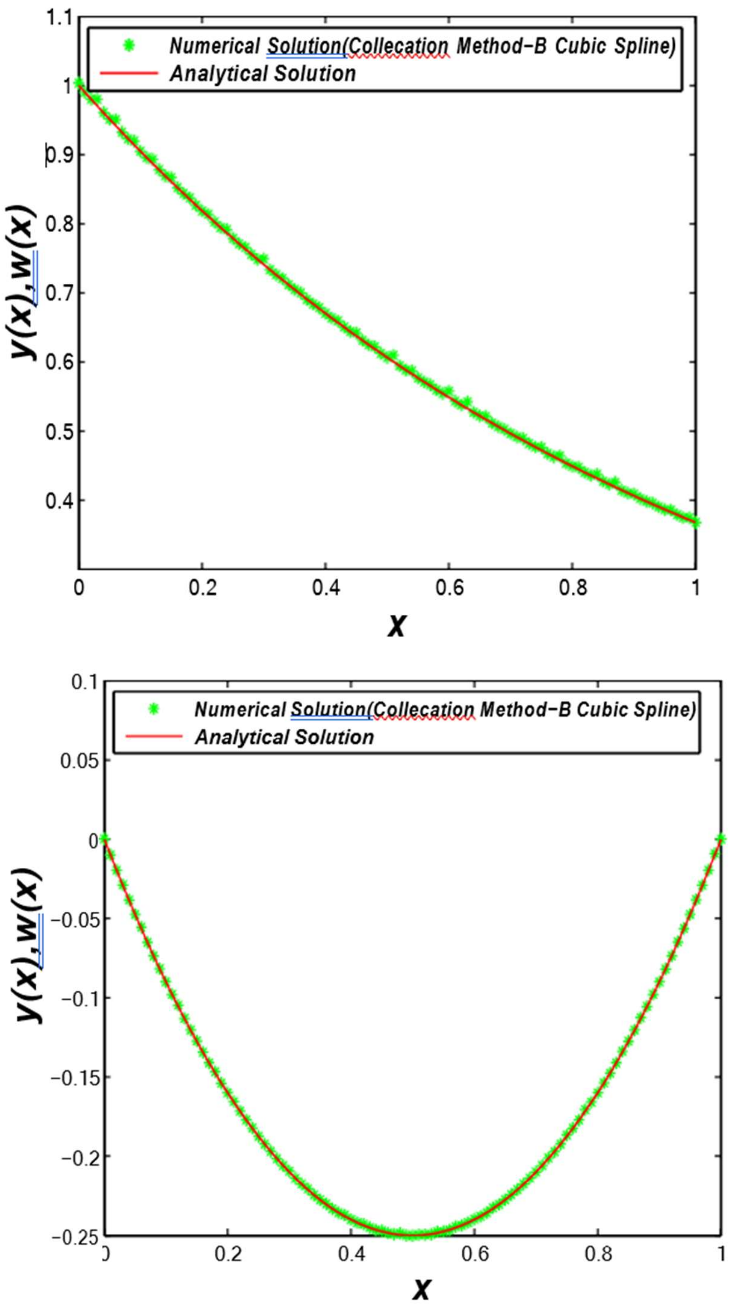
Exact solution: $u(\varsigma) = \varsigma(\varsigma - 1).$

Table 2: Absolute Errors of Solution of Quasi Numerical Linear F.D.E using compound method of spline and Collocation for Example 6.1

x	Numerical Solution	Absolute Errors
0.0	1.000000000000000	0
0.1	0.904817594804227	0.000019823231732
0.2	0.818712946229494	0.000017806848488
0.3	0.740814348309344	0.000003872372374
0.4	0.670318191658916	0.000001854376724
0.5	0.606530023335270	0.000000636377363
0.6	0.548811387721292	0.000000248372735
0.7	0.496585292067633	0.000000011723776
0.8	0.449328926293987	0.000000037823235
0.9	0.406569644002910	0.000000015737689
1	0.367879386715949	0.000000054455493

Table 3: Absolute Errors of Solution of Quasi Numerical Linear F.D.E using compound method of spline and Collocation for Example 6.2

x	Numerical Solution	Absolute Error
0.0	0	0
0.1	-0.090018337710736	0.000018337710736
0.2	-0.160017806158730	0.000017806158730
0.3	-0.210003859587736	0.000003859587736
0.4	-0.240001894719564	0.000001894719564
0.5	-0.250000639080207	0.000000639080207
0.6	-0.240000256003756	0.000000256003756
0.7	-0.210000010121801	0.000000010121801
0.8	-0.160000038465631	0.000000038465631
0.9	-0.090000015863977	0.000000015863977
1	0	0



b)

Figure 2: The Comparisons Curves of Collection Method Against the Analytical Solutions for Exam- ples (a) 6.1, and (b) 6.2

Discussion and Conclusion

In this paper, we have studied the several classes of FDEs and B-cubic spline base. Then, the CM has been introduced. The main contribution of this paper is to propose a numerical method for solving the quasi-linear FDEs. The collocation approach takes into account a numerical solution of this type. Some of the examples in the implementations are used to compare the numerical and analytical solutions for the test examples. From these numerical results which show in Figure 2 and Tables 2-3, we conclude that the proposed method is agree well with the analytical solutions for Examples 6.1-6.2. However, the proposed method is efficient and accurate.

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