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A COMMON FIXED POINT IN FUZZY METRIC SPACES

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Abstract:
In this paper we establish a fixed point results using compatibility and weakly compatibility of six self-maps in fuzzy metric spaces. We will extend the result of Singh and Chouhan[13] for common fixed points in fuzzy metric space.

Introduction

The concept of Fuzzy sets was given by Zadeh [17]. Subsequently, several researchers in Analysis and Topology used it. The paper is dealt with the Fuzzy metric space defined by Kramosil and Michalek [11] and modified by George and Veeramani [5]. Grebiec [6] has proved fixed point results for Fuzzy metric space. In this connection, Singh and Chouhan [13] introduced the concept of compatible mappings in Fuzzy metric space and proved the common fixed point theorem. Vasuki [15] proved the fixed point theorems using the concept of R-weak commutativity of mappings for Fuzzy metric space.

Recently, Jungck and Rhoades [10] introduced the concept of weak compatible maps. The concept is most general among all the commutativity maps. For this, every pair of R-weakly commuting self maps is compatible and each pair of compatible self maps is weakly compatible but the converse is not true.

A fixed-point theorem for six self maps using the concept of weak compatibility and compatibility of pairs of self-maps in fuzzy metric space has been proved in this paper. The result of Singh and Chouhan [13] has been generalized.

For this, we need the following definitions and Lemmas.

2. Preliminaries

Definition 2.1[2] A binary operation *: [0, 1] x [0, 1] $\rightarrow$ [0, 1] is called a t-norm if $([0,1]*)$ is an abelian topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for a, b, c, d $\in [0, 1]$.

Examples of t-norms are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 2.2 [9] The 3-tuple (X, M, *) is said to be a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a Fuzzy set in $X^2 \times [0, \infty]$ satisfying the following conditions:

(FM-1) M (x, y, 0) = 0,

(FM-2) M (x; y, t) = 1 for all t > 0 if and only if x = y, (FM-3) M (x, y, t) = M (y, x, t),

(FM-4) M (x, y, t) * M (y, z, s) ≤ M (x, z, t + s),

(FM-5) M (x, y,.): $[0,\infty) \rightarrow$ [0, 1] is left continuous,

(FM-6) $\lim_{t\rightarrow\infty}$ M (x, y, t) = 1.

for all x, y, z $\in$ X and s, t > 0.

Note that M (x, y, t) can be considered as the degree of nearness between x and y with respect to t. We identify x = y with M (x, y, t) = 1 for all t > 0. The following example shows that every metric space induces a Fuzzy metric space.

Example 2.1 [2] Let (X, d) be a metric space. Define a * b = min {a, b} and M (x, y, t) = $\frac{t}{t+d(x,y)}$ for all x, y $\in$ X and all t > 0. Then (X, M, *) is a Fuzzy metric space. It is called the Fuzzy metric space induced by d.

Definition 2.3 [3] A sequence $\{x_n\}$ in a Fuzzy metric space (X, M, *) is said to be a Cauchy sequence if and only if for each ε > 0, t > 0, there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1-\varepsilon$ for all n, m> n_0.

The sequence $\{x_n\}$ is said to converge to a point x in X if and only if for each ε > 0, t > 0 there exists $n_0 \in N$ such that $M(x_n, x, t) > 1-\varepsilon$ for all n, m≥ n_0.

A Fuzzy metric space (X, M, *) is said to be complete if every Cauchy sequence in it converges to a point in it.

Proposition 2.1 [13]Self mappings A and S of a Fuzzy metric space (X, M, *) are compatible then they are weakly compatible.

Proof. Suppose Ap = Sp, for some p in X. Consider a constant sequence $\{p_n\}$ = p. Now, $\{Ap_n\} \rightarrow Ap$ and $\{Sp_n\} \rightarrow Sp\,(= Ap)$.

As A and S are compatible we have $M(ASp_n, SAp_n, t) \rightarrow 1$ for all t > 0 as $n \rightarrow \infty$. Thus ASp = SAp and we get that (A, S) is weakly compatible.

The following is an example of pair of self maps in a Fuzzy metric space which are weakly compatible but not compatible.

Example 2.2 [9] Let (X, M, *) be a Fuzzy metric space where X= [0, 2]. t-norm is defined by a * b = min $\{a,b\}$ for all $a,b \in$ [0, 1] and $M(x,y,t) = e^{\frac{|x-y|}{t}}$ for all x, y $\in$ X. Define self maps A and S on X as follows:

$$Ax = \begin{cases} 2-x & \text{if} \quad 0 \le x < 1 \\ 2 & \text{if} \quad 1 \le x \le 2 \end{cases} \qquad \text{And}$$

$$Sx = \begin{cases} x & \text{if} \quad 0 \le x < 1 \\ 2 & \text{if} \quad 1 \le x \le 2 \end{cases}$$

Taking $\quad x_n = 1 - \frac{1}{n}; \;\; n = 1, 23, \ldots$

Then $\quad x_n \to 1, x_n < 1 \;\text{ and }\; 2 - x_n > 1$ for all n.

Also $\quad Ax_n, Sx_n \to 1 \text{ as } n \to \infty.$ Now

$$M(ASx_n, SAx_n, t) = e^{\frac{|ASx_n - SAx_n|}{t}} \to e^{-\frac{1}{2}} \ne 1 \;\text{ as } n \to \infty.$$

Hence the pair (A, S) is not compatible. Also set of coincident points of A and S is [1, 2].

Now for any x $\in$ [1, 2], Ax = 8x = 2 and AS(x) = A (2) = 2 = S (2) = SA(x). Thus A and S are weakly compatible but not compatible.

From the above example, it is obvious that the concept of weak compatibility is more general than that of compatibility.

Lemma 2.1 [1] Let $\{x_n\}$ be a sequence in a Fuzzy metric space (X, M, *). If there exists a number k $\in$ (0, 1) such that $M(x_{n+2}, x_{n+1}, kt) \ge M(x_{n+1}, x_n, t)$ for all and $n \in N$. Then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 2.2 [13] Let (X, M, *) be a Fuzzy metric space. If there exists $k \in (0,1)$ such that for all $x, y \in X$.

$M(x, y, kt) \ge M(x, y, t)$ for all $t > 0$, then $x = y$.

Theorem 2.1 [13] Let A, B, S, T, P and Q is self maps on a complete Fuzzy metric space (X, M, *) with * is a continuous t-norm for all t > 0 satisfying the following conditions

(a) P(X) $\subseteq$ ST(X), Q(X) $\subseteq$ AB(X);

(b) AB = BA, ST = TS, PB = BP, QT = TQ;

(c) (P, AB) is compatible and (Q, ST) is weakly compatible;

(d) Either AB or P is continuous;

(e) There exists k $\in$ (0, 1) such that

$$M(Px, Qy, kt) \ge \min\{M(ABx, Px, t), M(STy, Qy, t),$$
$$M(STy, Px, \beta t), M(ABx, Qy, (2-\beta)t)$$

$$M(ABx, STy, t)\},$$

For all x, y $\in$ X, $\beta \in (0, 2)$ and t > 0. Then A, B, S, T, P and Q have a unique common fixed point in X.

3. Main Result.

Theorem 3.1 Let A, B, S, T, P and Q be self-maps of a complete fuzzy metric space (X, M, *) with * is a continuous t-norm for all $t > 0$ satisfying the following conditions

(a) $P(x) \subseteq ST(x), \; Q(x) \subseteq AB(x)$

(b) AB = BA, ST = TS, PB = BP, QT = TQ.

(c) (P, AB) is compatible, and (Q, ST) is weakly compatible

(d) Either AB or P is continuous.

(e) There exists $K \in (0,1)$ such that

$$M(Px, Qy, kt) \geq \min\Big\{M(Qy, STy, t), M(ABx, STy, t),$$

$$\frac{M(Px, ABx, t), M(Qy, STy, t)}{M(Px, Qy, t)}, M(Px, ABx, t)\Big\}$$

For all $x, y \in x$ and $t > 0$. Then A, B, S, T, P and Q have a unique common fixed Point in X.

Proof. Let $x_0 \in X$ from (a) $\exists\, x_1, x_2 \in X$ such that

$$Px_0 = STx_1 = y_0 \text{ and } Qx_1 = ABx_2 = y_1$$

Inductively, we can contract seq $\{x_n\}$ and $\{y_n\}$ in X such that

$Px_{2n} = STx_{2n+1} = y_{2n}$ and $Qx_{2n+1} = ABx_{2n+2} = y_{2n+1}$ n = 0, 1, 2, ...

I. put $x = x_{2n}, y = x_{2n+1}$ for t > 0 in (e) we get

$$M(Px_{2n}, Qx_{2n+1}, kt) \geq \min\{M(Qx_{2n+1}, STx_{2n+1}, t),$$
$$M(ABx_{2n}, STx_{2n+1}, t),$$
$$\frac{M(Px_{2n}, ABx_{2n}, t), M(Qx_{2n+1}, STx_{2n+1}, t)}{M(Px_{2n}, Qx_{2n+1}, t)},$$
$$M(Px_{2n}, ABx_{2n}, t)\}$$

$$M(y_{2n}, y_{2n+1}, kt) \geq \min\{M(y_{2n}, y_{2n+1}, t),\; M(y_{2n-1}, y_{2n}, t)$$

$$\frac{M(y_{2n}, y_{2n-1}, t), M(y_{2n+1}, y_{2n}, t)}{M(y_{2n}, y_{2n+1}, t)},$$

$M(y_{2n}, y_{2n-1}, t)\}$ By (FM 3)

$$M(y_{2n}, y_{2n+1}, kt) \geq \min\{M(y_{2n+1}, y_{2n}, t),\ M(y_{2n-1}, y_{2n}, t)$$

$$\frac{M(y_{2n}, y_{2n-1}, t), M(y_{2n+1}, y_{2n}, t)}{M(y_{2n+1}, y_{2n}, t)},$$

$M(y_{2n-1}, y_{2n}, t)\}$

$$\geq \min\{M(y_{2n}, y_{2n+1}, t), M(y_{2n-1}, y_{2n}, t),$$

$M(y_{2n-1}, y_{2n}, t), M(y_{2n-1}, y_{2n}, t)\}$

$\geq \min\{M(y_{2n}, y_{2n+1}, t), M(y_{2n-1}, y_{2n}, t)\}$

$$M(y_{2n}, y_{2n+1}, kt) \geq \min\{M(y_{2n-1}, y_{2n}, t),\ M(y_{2n}, y_{2n+1}, t)\}$$

Similarly

$$M(y_{2n+1}, y_{2n+2}, kt) \geq \min\{M(y_{2n}, y_{2n+1}, t),\ M(y_{2n+1}, y_{2n+2}, t)\}$$

Therefore for all n even or odd, we have

$$M(y_n, y_{n+1}, kt) \geq \min\{M(y_{n-1}, y_n, t),\ M(y_n, y_{n+1}, t)\}$$

$$M(y_n, y_{n+1}, t) \geq \min\{M(y_{n-1}, y_n, t/k),\ M(y_n, y_{n+1}, t/k)\}$$

Be repeating the above inequality, we have

$$M(y_n, y_{n+1}, t) \geq \min\{M(y_{n-1}, y_n, t/k), M(y_n, y_{n+1}, t/k^m)\}$$

Since $M(y_n, y_{n+1}, t/k^m) \to 1$ as $m \to \infty$, it follows that

$$M(y_n, y_{n+1}, t) \geq M(y_{n-1}, y_n, t/k)$$

i.e. $M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t)$ for all $n \in N$ $t > 0$

By lemma 2.1, this implies that $\{y_n\}$ is Cauchy sequence in x. Since x is complete $\{y_n\} \to z \in X$. Also its subsequences converge to the same point i.e. $z \in X$.

$\{Qx_{2n+1}\} \to z$ and $\{STx_{2n+1}\} \to z$

$\{Px_{2n}\} \to z$ and $\{ABx_{2n}\} \to z$

Firstly, suppose AB is continuous.

As AB is continuous, $(AB)^2 x_{2n} \to ABz$ and $(AB)Px_{2n} \to ABz$

As (P, AB) is continuous pair, we have $P(AB)x_{2n} \to ABz$

II. Putting $x = ABx_{2n}, y = x_{2n+1}$ in (e), we have

$$M(PABx_{2n}, Qx_{2n+1}, kt) \geq \min\{M(Qx_{2n+1}, STx_{2n+1}, t), M(ABABx_{2n}, STx_{2n+1}, t), \frac{M(PABx_{2n}, ABABx_{2n}, t), M(Qx_{2n+1}, ST_{2n+1}, t)}{M(PABx_{2n}, Qx_{2n+1}, t)}, M(PABx_{2n}, ABABx_{2n}, t)\}$$

Letting $n \to \infty$, we get

$$\geq \min\{M(z, z, t), M(ABz, z, t), \frac{M(ABz, ABz, t), M(z, z, t)}{M(ABz, z, t)}, M(ABz, ABz, t)\}$$

$$\geq \min\left\{M(ABz, z, t), \frac{1}{M(ABz, z, t)}\right\}$$

$$M(ABz, z, kt) \geq M(ABz, z, t)$$

Therefore by lemma 2.2 $ABz = z$

(III) Putting $x = z$, $y = x_{2n+1}$ in (e) we get

$$M(Pz, Qx_{2n+1}, kt) \geq \min\{M(Qx_{2n+1}, STx_{2n+1}, t), M(ABz, STx_{2n+1}, t) \frac{M(Pz, ABz, t), M(Qx_{2n+1}, STx_{2n+1}, t)}{M(Pz, Qx_{2n+1}, t)}, M(Pz, ABz, t)\}$$

Letting $n \to \infty$, we get

$$M(Pz, z, kt) \geq \min\{M(z, z, t), M(ABz, z, t) \frac{M(Pz, ABz, t), M(z, z, t)}{M(Pz, z, t)}, M(Pz, ABz, t)\}$$

$$M(Pz, z, kt) \geq \min\{M(z, z, t), M(z, z, t) \frac{M(Pz, z, t), M(z, z, t)}{M(Pz, z, t)}, M(Pz, z, t)\}$$

$$M(Pz, z, kt) \geq M(Pz, z, t)$$

Hence by lemma 2.2 $Pz = z$

therefore $ABz = Pz = z$

IV. Put $x = Bz$, $y = x_{2n+1}$ in (e) we get

$$M(PBz, Qx_{2n+1}, kt) \geq \min\{M(Qx_{2n+1}, STx_{2n+1}, t), \ M((AB)Bz, STx_{2n+1}, t)$$
$$\frac{M(PBz, ABBz, t), M(Qx_{2n+1}, STx_{2n+1}, t)}{M(PBz, Qx_{2n+1}, t)}, \ M(PBz, ABBz, t)\}$$

As AB = BA, PB = BP, Letting $n \to \infty$, we get

$$M(PBz, z, kt) \geq \min\{M(z, z, t), \ M(B(AB)z, z, t)$$
$$\frac{M(BPz, B(AB)z, t), M(z, z, t)}{M(Bz, z, t)}, \ M(Bz, Bz, t)\}$$

$$M(Bz, z, kt) \geq \min\left\{M(Bz, z, t), \frac{1}{M(Bz, z, t)}\right\} \quad M(Bz, z, kt) \geq M(Bz, z, t)$$

From by lemma 2.2 Bz = z As ABz = z $\Rightarrow$ Az = z

Therefore Az = Bz = Pz = z

V. As $P(X) \subseteq ST(X)$, there exist $v \in X$ such that z = Pz = STv

Putting $x = x_{2n}$, y = v in (e) we get

$$M(Px_{2n}, Qv, kt) \geq \min\{M(Qv, STv, t), \ M(ABx_{2n}, STv, t)$$
$$\frac{M(Px_{2n}, ABx_{2n}, t), M(Qv, STv, t)}{M(Px_{2n}, Qv, t)},$$
$$M(Px_{2n}, ABx_{2n}, t)\}$$

Letting $n \to \infty$ we get

$$M(z, Qv, kt) \geq \min\{M(Qv, z, t), \ M(z, z, t)\frac{M(z, z, t), M(Qv, z, t)}{M(z, Qv, t)}, M(z, z, t)\}$$ (by FM3)

$$M(z, Qv, kt) \geq M(z, Qv, t)$$

By lemma 2.2 $Qv = z$. Hence STv = z = Qv

As (Q, ST) is weakly compatible we have STQv=QSTv

Thus STz = Qz

VI. Put $x = x_{2n}$, y = z in (e) we get

$$M(Px_{2n}, Qz, kt) \geq \min\{M(Qz, STz, t), \ M(ABx_{2n}, STz, t)$$

$$\frac{M(Px_{2n}, ABx_{2n}, t), M(Qz, STz, t)}{M(Px_{2n}, Qz, t)},$$

$M(Px_{2n}, ABx_{2n}, t)\}$

Letting $n \to \infty$ we get

$$M(z, Qz, kt) \geq \min\{M(Qz, Qz, t),\ M(z, Qz, t) \quad , \quad \frac{M(z, z, t), M(Qz, Qz, t)}{M(z, Qz, t)},$$

$M(z, z,, t)\}$

$$M(z, Qz, kt) \geq \min\left\{M(z, Qz, t), \frac{1}{M(z, Qz, t)}\right\}$$

$$M(z, Qz, kt) \geq M(z, Qz, t)$$

By Lemma 2.2 Qz = z = STz

VII. Put $x = x_{2n}$, y = Tz in (e) we get

$$M(Px_{2n}, QTz, kt) \geq \min\{M(Qz, STz, t),\ M(ABx_{2n}, STTz, t)$$

$$\frac{M(Px_{2n}, ABx_{2n}, t), M(QTz, STTz, t)}{M(Px_{2n}, QTz, t)}, M(Px_{2n}, ABx_{2n}, t)\}$$

Letting $n \to \infty$ and using condition (b) we get

$$M(Px_{2n}, QTz, kt) \geq \min\{M(z, z, t),\ M(z, Tz, t), \quad \frac{M(z, z, t), M(Tz, Tz, t)}{M(z, Tz, t)},$$

$M(z, z, t)\}$

$$M(z, Tz, kt) \geq \min\left\{M(z, Tz, t), \frac{1}{M(z, Tz, t)}\right\}$$

$$M(z, Tz, kt) \geq M(z, Tz, t)$$

From Lemma 2.2 Tz = z As STz = z $\Rightarrow$ Sz = z

Qz = Sz = Tz = z ... (B)

From (A) and (B) Az = Bz = Sz = Tz = Qz = Pz = z

Hence z is a common fixed point of A, B, S, T, P and Q.

Uniqueness.

Let u be another common fixed pt of A, B, S. T, P and Q Then

Au = Bu = Pu = Qu = Su = Tu = u

Put x = z, y = u in (e)

$$M(Pz, Qu, kt) \geq \min\left\{M(Qu, STu, t),\quad M(ABz, STu, t)\right.$$

$$\left.\frac{M(Pz, ABz, t), M(Qu, STu, t)}{M(Pz, Qu, t)},\quad M(Pz, ABz, t)\right\}$$

$$M(z, u, kt) \geq \min\left\{M(u, u, t),\ M(z, u, t)\ \frac{M(z, z, t), M(u, u, t)}{M(z, u, t)},\ M(z, z, t)\right\}$$

$$M(z, u, kt) \geq \min\left\{M(z, u, t), \frac{1}{M(z, u, t)}\right\}$$

$$M(z, u, kt) \geq M(z, u, t)$$

By Lemma 2.2 z = u

Hence z is a unique common fixed pt of self-maps A, B, S, T, P and Q.

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