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FIXED POINT THEOREMS IN CAT(0) SPACES USING NOOR ITERATIONS

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Abstract: In this article, we present strong convergence theorems for the Noor iteration method with quasi - nonexpansive multi - valued mappings in the context of CAT(0) spaces. Some similar results finding in the literature are expanded upon and improved by our findings.
Keywords : CAT(0) space, Noor iterations, Banach space, strong convergence, quasi - nonexpansive multimaps, multi-valued nonexapansive map.
MSC : 47H09, 47H10.

1. Introduction : One of the most significant topics in the set - valued analysis is the fixed point theory for multi – valued mappings. Banach and Schauder fixed point theorems as well as other well-known single-valued mappings fixed point theorems have been extended to multi-valued mappings in Banach spaces.

Nadler(6) changed the need of Banach single-valued mappings to multi-vlaued mappings in 1969. Numerous spectacular fixed point findings of various multi-valued mappings have recently been investigated in a range of scenarios (1,7,8,9,).Multi-valued mappings have several uses, including in the fields of game theory, differential inclusion, optimal control theory, and other aspects of physics.

Before working on this research we are read the literature of S. Ladsungnern, P. Kingkam, J.Nantadilok (1), S. Dhompongsa (2), L. Leustean (4), B. Panyanak (7), K.P.R. Sastry and G.V.R. Babu (8), N. Shahzad and H. Zegeye (9), Y. Song and H. Wang (11). In this article we are using Noor iteration technique for extended the result of S. Ladsungnern, P. Kingkam, J. Nantadilok.

Suppose that X : = (X, ‖ . ‖) is a Banach space and □□be a nonempty convex subset of X. The set of □□be called proximinal iff x Є X, and ∃ y Є □□such that ‖ x - y ‖ = d(x, □), where d(x, □) = inf { ‖ x - z ‖ : z Є □□}. The entire article the symbols are CB (□) and P(□) introduce to the family of nonempty proximinal bounded subset and nonempty closed bounded subset of □ correspondingly. For any a, b Є CB (□), define the metric-

H_d : CB (□) x CB (□) ⟶ r^+ by

H_d (a, b) = max {sup $_{x \in a}$ d (x, b), sup $_{x \in b}$ d (y, a) }

We say such H_d the Hausdorff metric on CB (□). Here, suppose $r^+ = [\, 0, \infty\,)$.

Definition: 1.1. Suppose t : □□⟶□□□be single-valued mapping. Thus t is said to the nonexpansive, if ‖ t (x) – t (y) ‖ ≤ ‖ x - y ‖, ∀ x, y Є □□

Definition: 1.2. Suppose □□: □□⟶□ CB (□) is a multi-valued mapping. Thus □□is said to be nonexpansive, if H_d (□ (x), □ (y)) ≤ ‖ x - y ‖, ∀ x, y Є □□

Definition: 1.3. [10] Suppose □□: □□⟶□ CB (□) is a multi-valued mapping. Thus □□is said to be quasi-nonexpansive, if F(□) ≠ ∅ and H_d (□ (x), p) ≤ ‖ x - p ‖ ∀ x Є □□and p Є F(□), where F(□) is the set of fixed point of multi-valued mapping □□

Definition: 1.4. [mann 5] assume t : □□⟶□□□be single-valued mapping. The Mann iteration technique begning to x_0 Є □□□be the sequence $\{x_n\}$ detailed from the following –

$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n tx_n$, $\alpha_n \in [0,1],\ n \geq 0$,

Where α_n fulfils a specific criteria .

Definition: 1.5. [Ishikawa 3] assume that t : □□⟶□□□be single-valued mapping. The Ishikawa iteration method, begning to x_0 Є □□□be the sequence $\{x_n\}$ detailed from the following –

$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n tx_n$, $\alpha_n \in [0,1],\ n \geq 0$,

$y_n = (1 - \beta_n)x_n + \beta_n tx_n$, $\beta_n \in [0,1],\ n \geq 0$,

Where α_n and β_n fulfils a specific criteria.

Definition: 1.6.[Noor] assume that t : □□⟶□□□be single-valued mapping. The Noor iteration method, begning to x_0 Є □□□be the sequence $\{x_n\}$ detailed from the following –

$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n tx_n$, $\alpha_n \in [0,1],\ n \geq 0$,

$y_n = (1 - \beta_n)x_n + \beta_n tx_n$, $\beta_n \in [0,1],\ n \geq 0$,

$z_n = (1 - \gamma_n)x_n + \gamma_n tx_n$, $\gamma_n \in [0,1],\ n \geq 0$,

Where α_n,β_n and γ_n fulfils a specific criteria.

A number of writers have looked into iterative methods that use Mann or Ishikawa iteration techniques to approximate the fixed points of nonexpansive single-valued mappings.

Mann and Ishikawa iteration methods for multi-valued mappings described by Sastry and Babu [8] as followings :

Definition: 1.7. [8] Assume □ : □□⟶□P□□□□a multi-valued mapping and p Є F(□).

i. The sequence of Mann iterates is defined by x_0 Є □□
$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n y_n$, α_n Є [0,1], $n \geq 0$, (1.1)
where y_n Є tx_n such that ‖ $y_n - p$ ‖ = d (p, tx_n)

ii. The sequence of Ishikawa iterates is defined by x_0 Є □□
$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n v_n$, α_n Є [0,1], $n \geq 0$,
$y_n = (1 - \beta_n)x_n + \beta_n w_n$, β_n Є [0,1], $n \geq 0$, (1.2)
where v_n Є ty_n such that ‖ $v_n - p$ ‖ = d (p, ty_n), and w_n Є tx_n such that ‖ $w_n - p$ ‖ = d (p, tx_n).

Sastry and Babu [8] showed that the Mann and Ishikawa iteration techniques for a multi-valued mapping t with fixed point p converges to q of t in the specific settings. And we are after study the literature of S. Ladsungnern, P. Kingkam, J. Nantadilok, Sastry and Babu, then describe Noor iteration method for our article is following –

iii. The sequence of Noor iterates is defined by x_0 Є □□
$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n v_n$, α_n Є [0,1], $n \geq 0$,
$y_n = (1 - \beta_n)x_n + \beta_n w_n$, β_n Є [0,1], $n \geq 0$,
$z_n = (1 - \gamma_n)x_n + \gamma_n u_n$, γ_n Є [0,1], $n \geq 0$, (1.3)

where u_n Є tz_n such that ‖ $u_n - p$ ‖ = d (p, tz_n), v_n Є ty_n such that ‖ $v_n - p$ ‖ = d (p, ty_n), and w_n Є tx_n such that ‖ $w_n - p$ ‖ = d (p, tx_n).

2. Preliminaries: we give some fundamental information about CAT(0) spaces and hyperbolic spaces in this part.

2.1 CAT(0) Spaces. Assume that (𝕏, d) be a metric space. A geodesic from x to y such that x,y Є 𝕏 is a map ω : [0, l] ⟶□X, [0, l] ⊂ R such that $\omega(0) = x$, $\omega(l) = y$, and d($\omega(t)$, $\omega(t')$) = ‖ t – t' ‖, for every t, t' Є [0, l]. More specifically, ω be an isometry and d(x, y) = l. Geodesic psegments linking x and y are used to describe the image of α and ω. The geodesic segment is identified by the symbols x and y when it is distinct. Any time two points in the space (X, d) are connected by a geodesic, the space is said to be geodesic. Additionally, if there is precisely one geodesic connecting x and y for any pair of x, y in 𝕏, the space is called to be specifically geodesic.

Definition 2.1: CAT(0) Space : Consider the geodesic space at (𝕏, d). It is a CAT(0) inequality if for any geographic triangle Δ ⊂ 𝕏. And x, y Є Δ, then we have $d(x, y) \leq d(\bar{x}, \bar{y})$, where $\bar{x}, \bar{y}$ Є $\bar{\Delta}$.

All complete, simply linked Riemannian manifolds with nonpositive sectional curvature are known to be CAT(0) space. Pre-Hilbret space, Euclidean buildings and many others are an illustration of CAT(0) spaces.

Definition 2.2: A geographic triangle $\Delta(p, q, r)$ in $(\mathbb{X}, d)$ is said to be CAT(0) space if for each $u, v \in \Delta(p, q, r)$ and for there comparision points $\overline{u}, \overline{v} \in \overline{\Delta}(\overline{p}, \overline{q}, \overline{r})$, one has

$d(u, v) \leq d_{R^2}(\overline{u}, \overline{v})$.

2.2 Hyperbolic Spaces : we review a few symbols for the hyperbolic metric space in this part. CAT(0) spaces are included in this class of spaces.

Definition 2.3 : [4] The hyperbolic space is triple $(\mathbb{X}, d, \mathbb{W})$ where $(\mathbb{X}, d)$ be a metric space and $\mathbb{W} : \mathbb{X} \times \mathbb{X} \times [0, 1] \longrightarrow \mathbb{X}$ is such that

W1. : $d(z, \mathbb{W}(x, y, \alpha)) \leq (1 - \alpha)d(z, x) + \alpha d(z, y)$,

W2. : $d(\mathbb{W}(x, y, \alpha), \mathbb{W}(x, y, \beta)) = |\alpha - \beta| d(x, y)$,

W3. : $\mathbb{W}(x, y, \alpha) = \mathbb{W}(y, x, (1- \alpha))$,

W4. : $d(\mathbb{W}(x, z, \alpha), \mathbb{W}(y, w, \alpha)) \leq (1 - \alpha)d(x, y) + \alpha d(z, w)$; $\forall\ x, y, z, w \in X$; $\alpha, \beta \in [0,1]$.

It follows to (W1) that, for each $x,y \in \mathbb{X}$ and $\alpha \in [0, 1]$,

$$d(x, \mathbb{W}(x, y, \alpha)) \leq \alpha d(x, y); \quad d(y, \mathbb{W}(x, y, \alpha)) \leq (1 - \alpha)\, d(x, y) \tag{2.1}$$

In fact, we have that,

$$d(x, \mathbb{W}(x, y, \alpha)) = \alpha d(x, y); \quad d(y, \mathbb{W}(x, y, \alpha)) = (1 - \alpha)\, d(x, y) \tag{2.2}$$

we can also write $\mathbb{W}(x, y, \alpha)$ in a hyperbolic space $(\mathbb{X}, d, \mathbb{W})$ using the notation $(1 - \alpha)x \oplus \alpha y$ for comparison between (2.2) and (2.1).

The family of Banach vector spaces or any normed vector space is an illustration of a hyperbolic spaces.

Lemma 2.4 : [2] suppose $(\mathbb{X}, d)$ is a CAT(0) space.

i. For $x,y \in \mathbb{X}$ and $\alpha \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that
$d(x, z) = \alpha d(x, y)$; and $d(y, z) = (1 - \alpha)\, d(x, y)$.
we take the symbol $(1 - \alpha)x \oplus \alpha y$ for the unique point z fulfilling (2.4) .
ii. For $x, y, z \in \mathbb{X}$ and $\alpha \in [0, 1]$ then we get,
$d((1 - \alpha)x \oplus \alpha y, z) \leq (1 - \alpha)d(x, z) + \alpha d(y, z)$.
iii. For $x, y, z \in \mathbb{X}$ and $\alpha \in [0, 1]$ then we get,
$d((1 - \alpha)x \oplus \alpha y, z)^2 \leq (1 - \alpha)d(x, z)^2 + \alpha d(y, z)^2 - \alpha(1 - \alpha)\, d(x, y)^2$.

3. Main Result: suppose (𝕏, d) is a CAT(0) space and □□be a nonempty convex subset of 𝕏 . In a similar vein, we present the definitions listed below.

Definition 3.1. A multi-valued mapping □ : □□⟶□P□□□□is said to be :

i. Nonexpansive if H_d (□x, □y) ≤ d (x, y) , ∀ x, y Є □□
ii. Quasi–nonexpansive if H_d (□x, □p) ≤ d (x, p), ∀ x Є □□and p Є F(□)

Following [9] ,we give the following definition

Definition 3.2. Suppose □□: □□⟶□ CB (□) is a multi-valued mapping. Let α_n, β_n Є [0,1], then the sequence of Ishikawa iterates is generated by x_0 Є □□

$$x_{n+1} = (1 - \alpha_n)x_n \oplus \alpha_n v_n, \quad \alpha_n \in [0,1],\ n \geq 0,$$

$$y_n = (1 - \beta_n)x_n \oplus \beta_n w_n, \quad \beta_n \in [0,1],\ n \geq 0, \qquad (3.1)$$

where v_n Є □y_n and w_n Є □x_n .

Following [1], we give the following definition

Definition 3.3. Suppose □□: □□⟶□ CB (□) is a multi-valued mapping. Let α_n, β_n, γ_n, Є [0, 1], then the sequence of Ishikawa iterates is generated by x_0 Є □□

$$x_{n+1} = (1 - \alpha_n)x_n \oplus \alpha_n v_n, \quad \alpha_n \in [0,1],\ n \geq 0,$$
$$y_n = (1 - \beta_n)x_n \oplus \beta_n w_n, \quad \beta_n \in [0,1],\ n \geq 0,$$
$$z_n = (1 - \gamma_n)x_n \oplus \gamma_n u_n, \quad \gamma_n \in [0,1],\ n \geq 0, \qquad (3.2)$$

where v_n Є □y_n, w_n Є □x_n and u_n Є □z_n .

Note that we substantially make use of lemma (2.4) [2] in the proof of our main results.

Lemma 3.4. Suppose □□be a nonempty closed convex subset of 𝕏, and be a CAT(0) space. Assume that F(□) ≠ ∅ and that □(p) = <p> for each p Є F(□). For the quasi-nonexpansive mapping □□: □□⟶□ CB (□). In this case let <x_n> represent the Noor iterations scheme produced by (3.2) consequently. For each p Є F(□), $\lim_{n\to\infty} d(x_n, p)$ exist.

Proof : Assume p Є F(□), then from (3.2) and using the lemma 2.4 (ii), then we get

$$d(z_n, p) = d((1 - \gamma_n)x_n \oplus \gamma_n u_n, p)$$

$$\leq (1 - \gamma_n)\, d(x_n, p) + \gamma_n\, d(u_n, p)$$

$$\leq (1 - \gamma_n)\, d(x_n, p) + \gamma_n H_d(\Box z_n, \Box p)$$

$$\leq (1 - \gamma_n)\, d(x_n, p) + \gamma_n\, d(x_n, p)$$

$$= (1 - \gamma_n + \gamma_n)\, d(x_n, p)$$

$$= d(x_n, p) \qquad 3.3$$

$$d(y_n, p) = d((1 - \square_n)x_n \oplus \square_n w_n, p)$$

$$\leq (1 - \square_n)\, d(x_n, p) + \square_n\, d(w_n, p)$$

$$\leq (1 - \square_n)\, d(x_n, p) + \square_n H_d(\square x_n, \square p)$$

$$\leq (1 - \square_n)\, d(x_n, p) + \square_n\, d(x_n, p)$$

$$= (1 - \square_n + \square_n)\, d(x_n, p)$$

$$= d(x_n, p) \qquad 3.4$$

$$d(x_{n+1}, p) = d((1 - \square_n)x_n \oplus \square_n v_n, p)$$

$$\leq (1 - \square_n)\, d(x_n, p) + \square_n\, d(v_n, p)$$

$$\leq (1 - \square_n)\, d(x_n, p) + \square_n H_d(\square y_n, \square p)$$

$$\leq (1 - \square_n)\, d(x_n, p) + \square_n\, d(y_n, p)$$

$$\leq (1 - \square_n)\, d(x_n, p) + \square_n\, d(x_n, p)$$

$$= (1 - \square_n + \square_n)\, d(x_n, p)$$

$$= d(x_n, p) \qquad 3.5$$

For any $p \in F(\square)$, $\lim_{n\to\infty} d(x_n, p)$ exist since the sequence $< d(x_n, p)>$ is decreasing and confined below. Moreover $<x_n>$ is bounded.

Theorem 3.5. Suppose that □□is a nonempty closed convex subset of $\mathbb{X}$. And that $(\mathbb{X}, d)$ is a CAT(0) space. Assume that □□: □□⟶□ CB (□) be a quasi nonexpansive mapping where $F(\square) \neq \emptyset$ and that $\square(p) = <p>$ for any $p \in F(\square)$. In this case let $<x_n>$ represent the Noor iterations scheme produced by (3.2). Assume that □□cmpiles with condition (I) and $\square_n, \square_n, \gamma_n \in [a, b] \subset [0, 1]$. When this happens, $<x_n>$ strongly approaches a □□fixed point.

Proof. Assume $p \in F(\square)$. So, since $<x_n>$ is bounded in the proof of lemma 3.4, then the sequences $<y_n>$ and $<z_n>$ are also bounded. As a result for all $n \geq 0$, there is $R > 0$ such that x, y and z are in $B_R(0)$. Using lemma 2.4 (iii), we have,

$$d(x_{n+1}, p)^2 = d((1 - \square_n)x_n \oplus \square_n v_n, p)^2$$

$$\leq (1 - \square_n)\, d(x_n, p)^2 + \square_n\, d(v_n, p)^2 - \square_n(1 - \square_n)\, d(x_n, v_n)^2$$

$$\leq (1 - \square_n)\, d(x_n, p)^2 + \square_n\, \mathbb{H}_d^2(\square y_n, \square p) - \square_n(1 - \square_n)\, d(x_n, v_n)^2$$

$$\leq (1 - \square_n)\, d(x_n, p)^2 + \square_n\, d(y_n, p) - \square_n(1 - \square_n)\, d(x_n, v_n)^2$$

$$\leq (1 - \square_n)\, d(x_n, p)^2 + \square_n\, d(y_n, p)^2 \qquad 3.6$$

$$d(y_n, p)^2 = d((1 - \square_n)x_n \oplus \square_n w_n, p)^2$$

$$\leq (1 - \square_n)\, d(x_n, p)^2 + \square_n\, d(w_n, p)^2 - \square_n(1 - \square_n)\, d(x_n, w_n)^2$$

$$\leq (1 - \square_n)\, d(x_n, p)^2 + \square_n \mathbb{H}_d^2\, (\square x_n, \square p) - \square_n(1 - \square_n)\, d(x_n, w_n)^2$$

$$\leq (1 - \square_n)\, d(x_n, p)^2 + \square_n\, d(x_n, p)^2 - \square_n(1 - \square_n)\, d(x_n, w_n)^2$$

$$= d(x_n, p)^2 - \square_n(1 - \square_n)\, d(w_n, x_n)^2 \qquad 3.7$$

$$d(z_n, p)^2 = d((1 - \gamma_n)x_n \oplus \gamma_n u_n, p)^2$$

$$\leq (1 - \gamma_n)\, d(x_n, p)^2 + \gamma_n\, d(u_n, p)^2 - \gamma_n(1 - \gamma_n)\, d(x_n, u_n)^2$$

$$\leq (1 - \gamma_n)\, d(x_n, p)^2 + \gamma_n \mathbb{H}_d^2\, (\square z_n, \square p) - \gamma_n(1 - \gamma_n)\, d(x_n, u_n)^2$$

$$\leq (1 - \gamma_n)\, d(x_n, p)^2 + \gamma_n\, d(x_n, p)^2 - \gamma_n(1 - \gamma_n)\, d(u_n, x_n)^2$$

$$= d(x_n, p)^2 - \gamma_n(1 - \gamma_n)\, d(u_n, x_n)^2 \qquad 3.8$$

Then from inequalities (3.6), (3.7), and (3.8) we get following inequality –

$d(x_{n+1}, p)^2 - d(y_n, p)^2 - d(z_n, p)^2 = (1 - \square_n)\, d(x_n, p)^2 + \square_n\, d(y_n, p)^2 - d(x_n, p)^2 + \square_n(1 - \square_n)\, d(w_n, x_n)^2 - d(x_n, p)^2 + \gamma_n(1 - \gamma_n)\, d(u_n, x_n)^2$

$= \square_n(1 - \square_n)\, d(w_n, x_n)^2 + \gamma_n(1 - \gamma_n)\, d(u_n, x_n)^2$

Thus we get,

$$d(x_{n+1}, p)^2 - d(z_n, p)^2 \leq \square_n(1 - \square_n)\, d(w_n, x_n)^2 + \gamma_n(1 - \gamma_n)\, d(u_n, x_n)^2$$

hence this implies that

$\sum_{n=1}^{m} \square_n(1 - \square_n)\, d(w_n, x_n)^2 + \gamma_n(1 - \gamma_n)\, d(u_n, x_n)^2 \leq d(x_{n+1}, p)^2 - d(z_n, p)^2 < \infty$. $\forall\ m \geq 1$.

So, $\sum_{n=1}^{m} \square_n(1 - \square_n)\, d(w_n, x_n)^2 + \gamma_n(1 - \gamma_n)\, d(u_n, x_n)^2 < \infty$.

Thus,

$$\lim_{n\to\infty} d(w_n, x_n)^2 = \lim_{n\to\infty} d(u_n, x_n)^2 = 0$$

Since, d is continuous . we have

$$\lim_{n\to\infty} d(w_n, x_n) = \lim_{n\to\infty} d(u_n, x_n) = 0$$

Also,

$$d(x_n, \square x_n) \leq d\,(x_n, w_n) \text{ and } d(x_n, \square x_n) \leq d\,(x_n, u_n) \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Since, □□satisfies condition (I) then we have ,

$$H(d(x_n, F(\square)\,)\,) \leq d(x_n, \square x_n) \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Then ,

$$\lim_{n\to\infty} d(x_n, F(\square)) = 0.$$

Thus there is a subsequence $< x_{n_k} >$ of $< x_n >$ such that $d(\, x_{n_k}, p_k\,) < \frac{1}{2^k}$ for some $< p_k > \subset F(\square)$ $\forall$ k. Note in the proof of lemma 3.4 we obtain,

$$d(\, x_{n_{k+1}}, p_k\,) \leq d(\, x_{n_k}, p_k\,) < \frac{1}{2^k}$$

we will show that $< p_k >$ is a Cauchy sequence in □. Notice that

$$d(\, p_{k+1},\; p_k) \leq d(\, p_{k+1},\; x_{n_k+1}) + d(\, x_{n_k+1}, p_k\,)$$

$$< \frac{1}{2^{k+1}} + \frac{1}{2^k}$$

$$< \frac{1}{2^{k-1}}\,.$$

This shows that $<p_k>$ is a Cauchy sequence in □□and thus converges to q Є □□

Since ,

$$d(p_k, \square q) \leq H_d\,(\square p_k, \square q)$$

$$\leq d(p_k, q)$$

And $p_k \longrightarrow q$ as $k \longrightarrow \infty$, It follows that d(q,□□q) = 0 and thus q Є F(□) and $< x_{n_k} >$ converges strongly to q. Since $\lim_{n\to\infty} d(x_n, q)$ exists, it follows that $< x_n >$ converges strongly to a fixed point q of □□□

This completes our proof.

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