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Ω - TRANSFORM OF WEIGHTED WAVE PACKET FRAMES FOR $L^2(\mathbb{R})$

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ABSTRACT. The purpose of this paper is to propose some type of transform $\Omega = (\omega_{\rho,\sigma,\tau,\eta,\mu,\nu})$ and for the weighted wave packet coefficient we have both necessary and sufficient conditions by applying Ω on $g \in L^2(\mathbb{R})$.

1. INTRODUCTION

A frame for Hilbert spaces was introduced by Duffin and Schaeffer [5]. A system $\{g_k\}$ is called *frame* for Hilbert spaces $\mathscr{H}$ if there exists constants $0 < \alpha \leq \beta < \infty$ such that

$$\alpha\|g\|^2 \leq \sum_{i=1}^{\infty} |\langle g, g_i\rangle|^2 \leq \beta\|g\|^2, \text{ for all } g \in \mathscr{H}. \tag{1.1}$$

Here, $\alpha > 0$ is called *lower* frame bound and $\beta > 0$ is called *upper* frame bound. The upper inequality in (1.1) holds for $\mathscr{H}$ then $\{g_i\}$ is called a Bessel sequence. The best bound for frame is defined by

$$\alpha_0 = \inf\{\beta : \beta > 0 \text{ satisfy } (1.1)\}$$
$$\beta_0 = \sup\{\alpha : \alpha > 0 \text{ satisfy } (1.1)\}$$

In the case $\alpha = \beta$ and $\alpha = \beta = 1$ then $\{g_k\}$ would be called *tight frame* and *normalized tight frame* for $\mathscr{H}$ respectively. The *synthesis operator* $\mathscr{T} : \ell^2 \to \mathscr{H}$ given by

$$\mathscr{T}(\{d_i\}) = \sum_{i=1}^{\infty} d_i g_i, \{d_i\} \in \ell^2$$

of the frame. The *analysis operator* is the adjoint $\mathscr{T}^* : \mathscr{H} \to \ell^2$ defined by

$$\mathscr{T}^*(g) = \{\langle g, g_i\rangle\}.$$

The *frame operator* is the composition of $\mathscr{T}$ and $\mathscr{T}^*$; $\mathscr{U} = \mathscr{T}\mathscr{T}^* : \mathscr{H} \to \mathscr{H}$ defined by

$$\mathscr{U}(g) = \sum_{i=1}^{\infty} \langle g, g_i\rangle g_i, g \in \mathscr{H}.$$

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and is the invertible, positive and continuous operator on $\mathscr{H}$. For each vector $g \in \mathscr{H}$ we have the expansion:

$$g = \mathscr{U}\mathscr{U}^{-1}g = \sum_{i=1}^{\infty} \langle \mathscr{U}^{-1}g, g_i \rangle g_i. \tag{1.2}$$

There is an unconditional convergence of the series given in (1.2) for all $g \in \mathscr{H}$ and the scalars $\langle \mathscr{U}^{-1}g, g_i \rangle$ are called *frame coefficients*. One may refer to [2, 3, 6, 12] for basic theory in frames.

The Wave Packet System was introduced by Cordoba and Fefferman [3] using dilation, modulation, and translation of Gaussian functions. In the past few years, several authors have studied wave packet systems, including [1, 4, 7, 8, 9, 10, 11].

We define and notate the terms and notations that will be used in this paper:

Assume g is a Lebesgue integrable function of complex value on $\mathbb{R}$ that is Banach space satisfying

$$\|g\|_p = \left(\int_{\mathbb{R}} |g(u)|^p du \right)^{\frac{1}{p}} < \infty,$$

where $1 \leq p < \infty$. The inner product on $L^2(\mathbb{R})$ is defined by

$$\langle g, h \rangle = \int_R g(u)\overline{h(u)}du,$$

where, the conjugate of h denoted by $\overline{h}$.

Here are the unitary operators on $L^2(\mathbb{R})$ defined by :

Translation $\leftrightarrow T_a g(u) = g(u-a)$, $a \in \mathbb{R}$.
Modulation $\leftrightarrow E_b g(u) = e^{2\pi i b t} g(u)$, $b \in \mathbb{R}$.
Dilation $\leftrightarrow D_a g(u) = \frac{1}{\sqrt{|a|}} g(\frac{u}{a})$, $a \neq 0, a \in \mathbb{R}$

For $a > 0$, $b, c \in \mathbb{R}$ and $g \in L^2(\mathbb{R})$, We know that

$$(D_{a_\eta} g)\hat{} = D_{a_\eta^{-1}} \hat{g}, (E_b g)\hat{} = T_b \hat{g}, (T_c g)\hat{} = E_{-c} \hat{g},$$

$$(D_{a_\eta} T_{b\mu} E_{c_\nu} g)\hat{} = D_{a_\eta^{-1}} E_{-b\mu} T_{c_\nu} \hat{g}.$$

2. Ω - TRANSFORM OF WEIGHTED WAVE PACKET FRAMES FOR $L^2(\mathbb{R})$

Definition 2.1. Assume $\phi \in L^2(\mathbb{R})$, $b \neq 0$, $\{c_\nu\}_{\nu \in \mathbb{Z}} \subset \mathbb{R}$ and $\{a_\eta\}_{\mu \in \mathbb{Z}} \subset \mathbb{R}^+$. A weighted wave packet system is one with the form $\{w_{\eta,\mu,\nu} D_{a_\eta} T_{b\mu} E_{c_\nu} \phi\}_{\eta,\mu,\nu \in \mathbb{Z}}$.

Definition 2.2. If a weighted wave packet system $\{w_{\eta,\mu,\nu} D_{a_\eta} T_{b\mu} E_{c_\nu} \phi\}_{\eta,\mu,\nu \in \mathbb{Z}}$ form a frame for $L^2(\mathbb{R})$, i.e., assume there exits constants $a_0 > 0$ and $b_0 > 0$, we have

$$a_0\|g\|^2 \leq \sum_{j\eta,\mu,\nu \in \mathbb{Z}} |\langle g, w_{\eta,\mu,\nu} D_{a_\eta} T_{b\mu} E_{c_\nu} \phi \rangle|^2 \leq b_0\|g\|^2, \text{ for all } g \in L^2(\mathbb{R}), \tag{2.1}$$

then we say that $\{w_{\eta,\mu,\nu} D_{a_\eta} T_{b\mu} E_{c_\nu} \phi\}_{\eta,\mu,\nu \in \mathbb{Z}}$ is a weighted wave packet frame.

The constant values $a_0 > 0$ and $b_0 > 0$ which refer to the *lower frame bound* and *upper frame bounds* for $\{w_{\eta,\mu,\nu} D_{a_\eta} T_{b\mu} E_{c_\nu} \phi\}_{\eta,\mu,\nu \in \mathbb{Z}}$, respectively . If upper inequality in (2.1) hold then the system $\{w_{\eta,\mu,\nu}D_{a_\eta}T_{b\mu}E_{c_\nu}\phi\}_{\eta,\mu,\nu\in\mathbb{Z}}$ is called the Weighted wave packet *Bessel sequence for* $L^2(\mathbb{R})$ with bound b_0.

For any function $\phi \in L^2(\mathbb{R})$, we consider the system of functions $\{\phi_{\eta,\mu,\nu}\}_{\eta,\mu,\nu\in\mathbb{Z}} \subset L^2(\mathbb{R})$ as

$$\{\phi_{\eta,\mu,\nu}(\varsigma) := w_{\eta,\mu,\nu}D_{a_\eta}T_{b\mu}E_{c_\nu}\phi(\varsigma) : \eta,\mu,\nu \in \mathbb{Z},\ \varsigma \in \mathbb{R}\} \tag{2.2}$$

When (2.2) is transformed using the Fourier transform, we get

$$\hat{\phi}_{\eta,\mu,\nu}(\xi) = w_{\eta,\mu,\nu}a_\eta^{-1/2}\hat{\phi}(a_\eta^{-1}\xi - c_\nu)e^{2\pi i\mu b a_\eta^{-1}\xi}.$$

The Plancheral theorem gives

$$d_{\eta,\mu,\nu} = \langle g, \phi_{\eta,\mu,\nu}\rangle = \int_{\mathbb{R}} g(\varsigma)\overline{\phi_{\eta,\mu,\nu}}(\varsigma)d\varsigma, \quad g \in L^2(\mathbb{R}) \tag{2.3}$$

The system defined in (2.2) is called a wave packet frame for $L^2(\mathbb{R})$ if there exists positive constants γ and δ such that

$$\gamma\|g\|^2 \le \sum_{\eta,\mu,\nu\in\mathbb{Z}} |\langle g, \phi_{\eta,\mu,\nu}\rangle|^2 \le \delta\|g\|^2, \text{ for all } g \in L^2(\mathbb{R}).$$

The constant values γ and δ which refer to the lower and upper frame bounds. If $\gamma = \delta = 1$ then for every function $g \in L^2(\mathbb{R})$ can be written as

$$g(\varsigma) = \sum_{\eta,\mu,\nu\in\mathbb{Z}} d_{\eta,\mu,\nu}\phi_{\eta,\mu,\nu}(\varsigma) \tag{2.4}$$

where $d_{\eta,\mu,\nu} = \langle g, \phi_{\eta,\mu,\nu}\rangle$ are given by (2.3), and it is called weighted wave packet coefficient for the expansion (2.4).

Definition 2.3. The Ω-transform of $\{s_{\eta,\mu,\nu}\}_{\eta,\mu,\nu\in\mathbb{Z}}$ for an infinite matrix $\Omega = (\omega_{\rho,\sigma,\tau,\eta,\mu,\nu})$ is defined by $\sum_{\eta,\mu,\nu\in\mathbb{Z}} \omega_{\rho,\sigma,\tau,\eta,\mu,\nu}s_{\eta,\mu,\nu}$.

Theorem 2.4. *Suppose an infinite matrix* $\Omega = (\omega_{\rho,\sigma,\tau,\eta,\mu,\nu})$ *whose elements are of the form* $e_{\rho,\sigma,\tau,\eta,\mu,\nu} = \langle \phi_{\rho,\sigma,\tau}, \phi_{\eta,\mu,\nu}\rangle$ *and for* $g \in L^2(\mathbb{R})$*, the following conditions hold*

(1) $\sum_{\eta,\mu,\nu} \phi_{\eta,\mu,\nu} \int_{\mathbb{R}} g(\xi)\overline{\phi_{\eta,\mu,\nu}(\xi)}d\xi = 1$
(2) $\lim_{\rho,\sigma,\tau\to\infty} \phi_{\rho,\sigma,\tau}(\varsigma) = 0.$

Then Ω*-transformation converges on* C_0 *for the weighted wave packet coefficients* $\{d_{\eta,\mu,\nu}\}$.

Proof. For $g \in L^2(\mathbb{R})$, considering that a matrix of infinite elements has the form $\langle \phi_{\eta,\mu,\nu}, \phi_{\rho,\sigma,\tau}\rangle$ and as defined in (2.3) , $\{d_{\eta,\mu,\nu}\}$ is the weighted wave packet coefficient. Then

$$\begin{aligned}\omega_{\rho,\sigma,\tau,\eta,\mu,\nu}d_{\eta,\mu,\nu} &= \langle \phi_{\eta,\mu,\nu}, \phi_{\rho,\sigma,\tau}\rangle\langle g, \phi_{\eta,\mu,\nu}\rangle \\ &= \int_{\mathbb{R}} \phi_{\eta,\mu,\nu}(\varsigma)\overline{\phi_{\rho,\sigma,\tau}}(\varsigma)d\varsigma \int_{\mathbb{R}} g(\varsigma)\overline{\phi_{\eta,\mu,\nu}}(\varsigma)d\varsigma \\ &= \int_{\mathbb{R}} g(\varsigma)\overline{\phi_{\rho,\sigma,\tau}}(\varsigma)d\varsigma \int_{\mathbb{R}} \phi_{\eta,\mu,\nu}(\varsigma)\overline{\phi_{\eta,\mu,\nu}}(\varsigma)d\varsigma.\end{aligned}$$

Thus,

$$\sum_{\eta,\mu,\nu\in\mathbb{Z}} \omega_{\rho,\sigma,\tau,\eta,\mu,\nu} d_{\eta,\mu,\nu} = \sum_{\eta,\mu,\nu\in\mathbb{Z}} \int_{\mathbb{R}} \int_{\mathbb{R}} \phi_{\rho,\sigma,\tau}(\varsigma) \overline{\phi_{\eta,\mu,\nu}}(\varsigma) g(\xi) \overline{\phi_{\eta,\mu,\nu}}(\xi) d\varsigma d\xi.$$

By (1) and (2), we get

$$\lim_{\rho,\sigma,\tau\to\infty} \sum_{\eta,\mu,\nu\in\mathbb{Z}} \omega_{\rho,\sigma,\tau,\eta,\mu,\nu} c_{\eta,\mu,\nu} = \lim_{\rho,\sigma,\tau\to\infty} \int_{\mathbb{R}} \phi_{\rho,\sigma,\tau}(\varsigma) d\varsigma = 0.$$

The proof is thus complete. □

Theorem 2.5. *Suppose that non-negative infinite matrix* $\Omega = (\omega_{\rho,\sigma,\tau,\eta,\mu,\nu})$ *with* $\sum_{\rho,\sigma,\tau\in\mathbb{Z}} \|\phi_{\rho,\sigma,\tau}\|^2 = 1$ *and let* $d_{\eta,\mu,\nu}$ *are coefficients associated with the weighted wave packet series expansion defined in* (2.4). *Then, for all* $g \in L^2(\mathbb{R})$ *we have*

$$\gamma_\phi \|g\|^2 \leq \sum_{\rho,\sigma,\tau\in\mathbb{Z}} |\langle \Omega g, \phi_{\rho,\sigma,\tau}\rangle|^2 \leq \delta_\phi \|g\|^2.$$

Here Ωg *signifies the* Ω*-transformation of* $g \in L^2(\mathbb{R})$, $0 < \gamma_\phi \leq \delta_\phi < \infty$.

Proof. Consider, $g(\varsigma) = \sum_{\eta,\mu,\nu\in\mathbb{Z}} \langle g, \phi_{\eta,\mu,\nu}\rangle \phi_{\eta,\mu,\nu}(\varsigma)$.

Applyimg Ω-transform on g, we obtain

$$\Omega g(\varsigma) = \sum_{\rho,\sigma,\tau\in\mathbb{Z}} \langle \Omega g, \phi_{\rho,\sigma,\tau}\rangle \phi_{\rho,\sigma,\tau}(\varsigma).$$

Therefore,

$$\begin{aligned} \sum_{\rho,\sigma,\tau\in\mathbb{Z}} |\langle \Omega g, \phi_{\rho,\sigma,\tau}\rangle|^2 &\leq \sum_{\rho,\sigma,\tau\in\mathbb{Z}} \int_{\mathbb{R}} |\Omega g(\varsigma)|^2 |\overline{\phi_{\rho,\sigma,\tau}(\varsigma)}|^2 d\varsigma \\ &\leq \|\Omega\|^2 \|g\|^2 \sum_{\rho,\sigma,\tau\in\mathbb{Z}} \|\phi_{\rho,\sigma,\tau}\|^2 . \end{aligned}$$

Thus, we have

$$\sum_{\rho,\sigma,\tau\in\mathbb{Z}} |\langle \Omega g, \phi_{\rho,\sigma,\tau}\rangle|^2 \leq \delta_\phi \|g\|^2, \tag{2.5}$$

where $\delta_\phi > 0$.

For each $g \in L^2(\mathbb{R})$, we get

$$f(\varsigma) = \left[\sum_{\rho,\sigma,\tau\in\mathbb{Z}} |\langle \Omega g, \phi_{\rho,\sigma,\tau}\rangle|^2\right]^{-\frac{1}{2}} g(\varsigma).$$

Clearly,

$$\langle \Omega f, \phi_{\rho,\sigma,\tau}\rangle = \left[\sum_{\rho,\sigma,\tau\in\mathbb{Z}} |\langle \Omega g, \phi_{\rho,\sigma,\tau}\rangle|^2\right]^{-\frac{1}{2}} \langle \Omega g, \phi_{\rho,\sigma,\tau}\rangle.$$

Thus,

$$\sum_{\rho,\sigma,\tau\in\mathbb{Z}} |\langle \Omega g, \phi_{\rho,\sigma,\tau}\rangle|^2 \leq 1.$$

Assume there exist constant $\lambda > 0$ such that $\|\Omega g\|^2 \leq \lambda$, then

$$\left[\sum_{\rho,\sigma,\tau\in\mathbb{Z}} |\langle \Omega g, \phi_{\rho,\sigma,\tau}\rangle|^2\right]^{-1} \|g\|^2 \leq \frac{\Omega}{\|\gamma\|^2} = \gamma_\phi > 0.$$

Therefore,

$$\gamma_\phi \|g\|_2^2 \leq \sum_{\rho,\sigma,\tau\in\mathbb{Z}} |\langle \Omega g, \phi_{\rho,\sigma,\tau}\rangle|^2. \tag{2.6}$$

From (2.5) and (2.6), gives

$$\gamma_\phi \|g\|^2 \leq \sum_{\rho,\sigma,\tau\in\mathbb{Z}} |\langle \Omega g, \phi_{\rho,\sigma,\tau}\rangle|^2 \leq \delta_\phi \|g\|^2 .$$

The proof is thus complete. □

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