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A NEW BLOCK INTEGRATOR FOR SOLVING THIRD-ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

A uniform order block method is formulated using collocation and interpolation techniques with Hermite polynomials as basis function for solving third order initial value problems in ordinary differential equations. The convergence and stability properties of the new method are investigated. The new method is tested on some numerical problems to demonstrate the applicability, accuracy and efficiency of the new method. *Keywords:* Multiderivative, Hermite Polynomials, Third Order Differential Equations, Block method.

INTRODUCTION

In this study, we considered the third order ordinary differential equation is of the form:

$$y''' = f(x,y,y',y''), y(a) = \alpha, \qquad y'(a) = \beta, \qquad y''(a) = \gamma, \tag{1}$$

Where f is a continuous differentiable function.

Equation (1) is traditionally solved by reducing the problem to an equivalent system of three first order ordinary differential equations. Jennings [4], Awoyemi [5], and Jator [10, 11] are few of the authors that discussed the technique of reducing *nth* order ordinary differential equation (1) to a system of equations. The rigour of writing computer subroutine-sub program within the main program to get starting values is a major drawback associated with methods that solve equivalent systems of first order ordinary differential equations. According to Awoyemi [5, 6, 7], the consequence of this is extra computational effort, and computer time and storage wastage. In view of these setbacks, methods that are based on reduction approach are inefficient and might not be very suitable in application. Therefore, there is a need to develop direct methods to handle high order ordinary differential equations. Adeyeye and Zurni [2], Anake et al [3], Kuboye et al [12], Awoyemi [8], Mohammed [19] among other authors have developed methods that solve problem (1) without having to reduce it to a system of equations.

These authors used various polynomials such as such as Legendre polynomials, Power series, Lucas polynomials, Taylor series and Chebyshev polynomials as basis functions for the formulation of block methods for the solution of problem (1). Linear Multistep Methods derived from continuous schemes constitutes block methods. Block methods perform better than predictor-corrector methods in terms of accuracy and efficiency.

The focus of this paper is to formulate a four step block method through interpolation and collocation techniques for solving third order initial value problems with the Hermite polynomials as the basis functions.

DERIVATION OF THE METHOD

In order to solve the initial value problem (1) in the interval $[a,b]$ based on the partition $a = x_0 \leq x_1 \leq \cdots x_n = b$ with a uniform step length of $h = x_n - x_{n-1}, n = 0, 1, \cdots, N-1$,
We consider an approximate solution $Y(x)$ to the analytical solution $y(x)$ of the form:

$$Y(x) = \sum_{r=0}^{7} a_r H_r(x - x_n), \qquad (2)$$

Where $H_r(x)$ is the Probabilist's Hermite polynomial function of degree r defined as follows:

$$H_r(x) = (-1)^r e^{\frac{x^2}{2}} \frac{d^r}{dx^r} e^{-\frac{x^2}{2}}. \qquad (3)$$

The Hermite polynomial (3) satisfies the recurrence relation:

$$H_{r+1}(x) = xH_r(x) - H'_r(x), \quad r \geq 1; \qquad (4)$$

With the initial conditions: $H_0(x) = 1$ and $H_1(x) = x$.

To determine the values of the coefficients a_r, $r = 0(1)7$, in equation (2), set

$$\sum_{r=0}^{7} a_r H_r(x - x_n) = y_{n+i}, i = 0, 1, 3, \qquad (5)$$

and

$$\sum_{r=0}^{7} a_r H'''_r(x - x_n) = f_{n+i}, i = 0(1)4, \qquad (6)$$

Where n is the grid index and $y_{n+i} = Y(x_{n+i})$.
Equations (5) and (6) provide a system of eight equations whose solution gives the values of the coefficients a_r, $r = 0(1)7$, which are substituted into (2) and after some algebraic manipulations yields a continuous scheme of the form:

$$Y(x) = \sum_{i=0}^{4} \alpha_i(x) y_{n+i} + h^3 \sum_{i=0}^{4} \beta_i(x) f_{n+i}, \qquad (7)$$

Where $\alpha_i(x)$, $\beta_i(x)$, $i = 0(1)4$ are continuous coefficients.
Evaluating (7) at $x = x_{n+2}, x_{n+4}$ to obtain:

$$y_{n+3} + 3y_{n+2} - 3y_{n+1} + y_n = \frac{h^3}{240}\Big(-f_n - 116f_{n+1} - 126f_{n+2} + 4f_{n+3} - f_{n+4}\Big)$$

$$y_{n+4} - 2y_{n+3} + 2y_{n+1} - y_n = \frac{h^3}{120}\Big(f_n + 56f_{n+1} + 126f_{n+2} + 56f_{n+3} + f_{n+4}\Big). \qquad (8)$$

(9)

Now, differentiating (7) in turn with respect to x gives

$$Y'(x) = \sum_{i=0}^{4} \alpha_i'(x) y_{n+i} + h^2 \sum_{i=0}^{4} \beta_i'(x) f_{n+i}, \qquad (10)$$

$$Y''(x) = \sum_{i=0}^{4} \alpha_i''(x) y_{n+i} + h^2 \sum_{i=0}^{4} \beta_i''(x) f_{n+i}, \qquad (11)$$

Evaluating equations (10) and (11) at $x = x_n, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}$ to get the complementary methods. However, equations (10), (11) and the complementary methods form the new block method which can be written in matrix form as:

$$AY_m = BY_{m-1} + h^3\Big[Cf_n + DF_m\Big]. \qquad (12)$$

That is,

$$\begin{pmatrix} -3 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -9 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & -1 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 9 & 0 & -5 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ 15 & 0 & -7 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 3 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 3 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 3 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 3 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ hy'_{n+1} \\ hy'_{n+2} \\ hy'_{n+3} \\ hy'_{n+4} \\ h^2y''_{n+1} \\ h^2y''_{n+2} \\ h^2y''_{n+3} \\ h^2y''_{n+4} \end{pmatrix}$$

$$
= \begin{pmatrix}
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -8 & 0 & 0 & 0 & -6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \\ hy'_{n-3} \\ hy'_{n-2} \\ hy'_{n-1} \\ hy'_n \\ h^2y''_{n-3} \\ h^2y''_{n-2} \\ h^2y''_{n-1} \\ h^2y''_n
\end{pmatrix}
$$

$$
+h^3 \begin{pmatrix}
\frac{-1}{240} \\ \frac{1}{120} \\ \frac{207}{560} \\ \frac{-43}{840} \\ \frac{5}{5040} \\ \frac{9}{280} \\ \frac{61}{1680} \\ \frac{-117}{120} \\ \frac{17}{240} \\ \frac{-1}{120} \\ \frac{9}{240} \\ \frac{-5}{120}
\end{pmatrix} f_n + h^3 \begin{pmatrix}
\frac{-116}{240} & \frac{-126}{240} & \frac{4}{240} & \frac{-1}{240} \\
\frac{56}{120} & \frac{126}{120} & \frac{56}{120} & \frac{1}{120} \\
\frac{1328}{560} & \frac{66}{560} & \frac{96}{560} & \frac{-17}{560} \\
\frac{-1172}{840} & \frac{-468}{840} & \frac{4}{840} & \frac{-1}{840} \\
\frac{-104}{5040} & \frac{-1482}{5040} & \frac{-104}{5040} & \frac{5}{5040} \\
\frac{524}{280} & \frac{1020}{280} & \frac{132}{280} & \frac{-5}{280} \\
\frac{6560}{1680} & \frac{14310}{1680} & \frac{10256}{1680} & \frac{733}{1680} \\
\frac{-361}{120} & \frac{30}{120} & \frac{-39}{120} & \frac{7}{120} \\
\frac{-76}{240} & \frac{-204}{240} & \frac{28}{240} & \frac{-5}{240} \\
\frac{135}{120} & \frac{126}{120} & \frac{-23}{120} & \frac{3}{120} \\
\frac{196}{240} & \frac{708}{240} & \frac{300}{240} & \frac{-13}{240} \\
\frac{151}{240} & \frac{222}{240} & \frac{473}{240} & \frac{119}{240}
\end{pmatrix}
\begin{pmatrix}
f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4}
\end{pmatrix}.
$$

ANALYSIS OF THE METHOD

Basic properties of the block method are considered and analyzed to establish the efficiency and reliability of the method. The following properties are analyzed: Order, error constant, consistence, Zero stability and Convergence.

Definition (Local Truncation Error)

According to Lambert [13], the local truncation error associated with the third order linear multistep method:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h^3 \sum_{i=0}^{k} \beta_i f_{n+i} \tag{13}$$

Is defned by the difference operator

$$L\Big[y(x):h\Big] = \sum_{i=0}^{k} \Big[\alpha_i y(x_n+ih) - h^3 \beta_i y'''(x_n+ih)\Big], \tag{14}$$

Where $y(x)$ is an arbitrary function, continuously differentiable on $[a,b]$. Expanding (14) in Taylor series about point x leads to the expression:

$$L\Big[y(x):h\Big] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \cdots + C_p h^p y^{(p)}(x) + \cdots C_{p+3} h^{p+3} y^{(p+3)}(x) \quad (15)$$

Where $C_0, C_1, C_2, \cdots, C_p, \cdots, C_{p+2}$ are obtained as follows:

$$C_0 = \sum_{i=0}^{k} \alpha_i \quad C_1 = \sum_{i=1}^{k} i\alpha_i \quad C_2 = \frac{1}{2!}\sum_{i=1}^{k} i^2 \alpha_i \quad \cdots$$

$$C_q = \frac{1}{q!}\Big[\sum_{i=1}^{k} i^q \alpha_i - q(q-1)(q-2)\sum_{i=1}^{k} \beta_i i^{q-3}\Big] \quad (16)$$

Therefore, the method (13) is of order p if

$C_0 = C_1 = C_2 = \cdots = C_p = C_{p+1} = C_{p+2} = 0$ and $C_{p+3} \neq 0$.

The constant $C_{p+3} \neq 0$ is called the error constant and $C_{p+3} h^{p+3} y^{(p+3)}(x)$ is the principal local truncation error at x_n. Using the above definition, the block method (12) is of order $p = 5$ and the error constant

$$C_{p+3} = \Big[\frac{1}{480}, \frac{1}{30240}, \frac{13}{560}, \frac{-1}{1120}, \frac{-623}{5000000}, \frac{1}{1120}, \frac{-13}{560}, \frac{59}{1344}, \frac{83}{6720}, \frac{1171}{2240}, \frac{83}{6720}, \frac{-59}{1344}\Big]^T. \quad (17)$$

Consistency

The linear multistep method (12) is of order $p = 5 \geq 1$. Hence, the new block method is consistent following Jator [11].

Zero Stability

Upon the normalization of equation (12), we obtain

$$A_0^* Y_m = A_1^* Y_{m-1} + h^3\Big[B_0^* f_n + B_1^* F_m\Big], \quad (18)$$

Where

$$A_0^* = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix} \quad A_1^* = \begin{pmatrix} 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

Now,

$$\rho(r) = det(rA_0^* - A_1) \text{ as } h \to 0. \quad (19)$$

Solving for r in the equation

$\rho(r) = r^{11}(r-1) = 0 \Rightarrow r = 0, 1.$

Therefore, the new block method is zero stable.

Convergence of the Method

The block method (12) is convergent since it satisfies the necessary and sufficient condition of consistency and zero stability following Henrici [9].

NUMERICAL EXAMPLES

To investigate the efficiency of the method, we apply the new method to solve some test problems.

Example 4.1

Solve the initial value problem:

$y''' + y' = 0;\ y(0) = 0, y'(0) = 1, \quad y''(0) = 2. \quad h = 0.1.$

Analytical Solution is $y(x) = 2(1-\cos x) + \sin x$.

Source: Anake et al [3].

Example 4.2

Solve the initial value problem:

$y''' = e^x; \quad y(0) = 3, \quad y'(0) = 1, \quad y''(0) = 5. \quad h = 0.1.$

Analytical Solution is $y(x) = 2 + 2x^2 + e^x$.

Source: Olabode and Yusuph [16].

Example 4.3

Solve the non linear initial value problem:

$y''' = y'(2xy'' + y');\ y(0) = 1, \quad y'(0) = 0.5, \quad y''(0) = 0. \quad h = 0.01.$

Analytical Solution is $y(x) = 1 + \frac{1}{2}\log\left(\frac{2+x}{2-x}\right).$

Source: Adoghe and Omole [17].

Example 4.4

Solve the linear initial value problem:

$y''' + 5y'' + 7y' + 3y = 0; y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -1. \quad h = 0.1.$

Analytical Solution is $y(x) = e^{-x} + xe^{-x}$.

Source: Sagir [18].

Example 4.5

Solve the linear initial value problem:

$y''' - y'' + y' - y = 0; y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -1. \quad h = 0.01.$

Analytical Solution is $y(x) = \cos x$.

Source: Sagir [18].

Absolute errors in the solutions of Examples 4.1, 4.2, 4.3, 4.4 and 4.5 based on the new method are respectively presented in Tables 1, 2, 3, 4 and 5 in comparison with existing methods in literature.

Table 1: Absolute Errors for Example 4.1

x	New Method	Awoyemi [5]	Olabode [15]	Anake et al [3]	Adeyefa [1]
0.1	6.1893E-11	–	1.6654E-08	1.6088E-09	2.3300E-10
0.2	3.9807E-10	8.8507E-07	3.8095E-07	1.0387E-08	1.4670E-09
0.3	9.6861E-10	–	1.5664E-07	2.9572E-08	4.800E-09
0.4	1.8043E-09	6.6921E-06	3.9865E-06	2.3147E-07	1.1230E-08
0.5	2.7321E-09	–	7.9597E-06	4.5420E-07	2.1767E-08
0.6	3.8183E-09	2.3718E-05	1.3680E-05	1.4746E-06	3.7500E-08
0.7	5.0297E-09	–	2.1195E-05	2.8734E-06	6.3733E-08
0.8	6.3766E-09	5.5181E-05	3.0396E-05	4.6826E-06	9.2767E-08
0.9	7.7337E-09	–	4.1008E-05	6.9217E-06	1.2910E-07
1.0	9.0876E-09	1.0338E-05	5.2605E-05	9.5974E-06	1.7573E-07

CONCLUSION

In this paper, we have constructed a direct 4 –step multiderivative integrator which is efficient and suitable for solving third order ordinary differential equations. The method has

Table 2: Absolute Errors for Example 4.2

x	New Method	Olabode and Yusuph [16]	Obarhua and Kayode [14]
0.1	4.0986E-11	7.5647E-11	4.6567E-11

0.2	2.6433E-10	1.8398E-09	4.2286E-10
0.3	6.4459E-10	4.4240E-09	1.5120E-09
0.4	1.2072E-09	1.0359E-08	3.7373E-09
0.5	1.8641E-09	1.1299E-08	1.3518E-08
0.6	2.7826E-09	1.4610E-08	1.3518E-08
0.7	3.9248E-09	2.0530E-08	2.2162E-08
0.8	5.3288E-09	1.9508E-08	3.4130E-08
0.9	6.8630E-09	1.0843E-08	5.0123E-08
1.0	8.7771E-09	1.5410E-08	7.0907E-08

Table 3: Absolute Errors for Example 4.3

x	New Method	Adoghe and Omole [17]
0.1	4.6655E-19	2.2204E-16
0.2	3.0159E-18	0.0000E+00
0.3	7.3211E-18	1.9984E-15
0.4	1.3713E-17	1.4211E-14
0.5	2.1632E-17	5.5511E-14
0.6	3.5882E-17	1.6409E-13
0.7	5.5473E-17	3.8813E-13
0.8	8.1610E-17	7.9980E-13
0.9	1.1047E-16	1.5010E-12
1.0	1.5037E-16	2.6241E-12

Shown acceptable solution and the method performed better and converges faster than some existing methods.

Table 4: Absolute Errors for Example 4.4

x	New Method	Sagir [18]
0.1	1.5505E-10	6.4300E-08
0.2	8.5291E-10	2.7200E-08
0.3	1.74460E-09	3.0500E-08
0.4	2.7843E-09	8.9800E-08
0.5	3.3548E-09	4.4260E-07
0.6	3.7430E-09	7.7260E-07
0.7	3.9305E-09	1.9523E-06
0.8	4.02890E-09	1.0274E-06
0.9	3.7452E-09	1.3509E-06
1.0	3.3243E-09	1.3470E-05

Table 5: Absolute Errors for Example 4.5

x	New Method	Sagir [18]	Adeyefa [1]
0.01	3.4648E-19	1.9990E-07	1.7460E-07
0.02	2.2405E-18	1.9560E-07	4.1500E-07
0.03	5.4885E-18	1.3651E-07	1.4021E-06
0.04	1.0313E-17	2.5210E-07	3.2914E-06
0.05	1.5886E-17	1.3039E-06	6.3483E-06

0.06	2.3064E-17	3.0280E-06	1.0822E-05
0.07	3.1654E-17	3.3453E-06	1.6945E-05
0.08	4.1878E-17	1.2405E-06	2.4934E-05
0.09	5.2909E-17	1.3290E-06	3.4989E-05
0.10	9.5609E-17	1.7180E-05	1.0525E-04

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