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GRAPH'S MONOPHONIC VERTEX COVERING NUMBER

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Abstract

For a connected graph G of order $n \geq 2$, a set S of vertices of G, is monophonic vertex cover of G if S is both a monophonic set and a vertex cover of G. The minimum cardinality of a monophonic vertex cover of G is called the monophonic vertex covering number of G and is denoted by m_α (G). Any monophonic vertex cover of cardinality m_α (G) is a m_α-set of G. Some general properties satisfied by monophonic vertex cover are studied. The monophonic vertex covering number of several classes of graphs are determined.

Keywords: monophonic set, vertex covering set, monophonic vertex cover, monophonic vertex covering number.

1. Introduction

By a graph G = (V, E), we mean a finite undirected simple connected graph. The order and size of G are denoted by n and m respectively. For basic graph theoretic terminology we refer to Harary[12]. The distance *d(u,v)* between two vertices u and v in a connected graph G is the length of a shortest u-v path in G[4]. For a vertex v of G, the eccentricity $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the radius, *rad G* and the maximum eccentricity is its diameter, *diam G*. The neighbourhood of a vertex v of G is the set $N(v)$ consisting of all vertices which are adjacent with v. A vertex v is a simplical vertex or an extreme vertex of G if the subgraph induced by its neighbourhood $N(v)$ is complete. A caterpillar is a tree of order 3 or more, the removal of whose end vertices produces a path called the spine of the caterpillar. A diametral path of a graph is a shortest path whose length is equal to the diameter of the graph. A tree containing exactly two non-pendent vertices is called a double star denoted by S_{k_1,k_2} where k_1 and k_2 are the number of pendent vertices on these two non-pendent vertices. A graph G is called triangle free if it does not contain cycles of length 3. A set of vertices no two of which are adjacent is called an independent set. By a matching in a graph G, we mean an independent set of edges of G. A maximal matching is a matching M of a graph G that is not a subset of any other matching. The independence number $\beta(G)$ of G is the maximum number of vertices in an independent set of vertices of G. A subset $S \subseteq V(G)$ is a dominating set if every vertex in $V-S$ is adjacent to at least one vertex in S. A set $S \subseteq V(G)$ is called a global dominating set if it is a dominating set of both G and $\bar{G}$ (the complement of G). The minimum cardinality of a dominating set in a graph G is called the dominating number of G and denoted by $\gamma(G)$. The dominating number is further studied in [1-3,10-11].

A geodetic set of G is a set S⊆V(G) such that every vertex of G is contained in a geodesic joining some pair of vertices in S. The geodetic number $g(G)$ of G is the minimum cardinality of its geodetic sets. The geodetic number of a graph was introduced in [6] and further studied in [5,7]. A subset $S \subseteq V(G)$ is called geodetic global dominating set of G if S is both geodetic and global dominating set of G. The geodetic global domination number of a graph was introduced in [15] and further studied in [16,17]. A chord of a path P is an edge joining two non-adjacent vertices of P. A path P is called a monophonic path if it is a chordless path. A set S of vertices of G is a monophonic set of G if each vertex v of G lies on an x-y monophonic path for some $x,y \in S$. The minimum cardinality of a monophonic set of G is the monophonic number of G and is denoted by m(G). Any monophonic set of cardinality m(G) is a minimum monophonic set or a monophonic basis or a $m-set$ of G. The monophonic number of a graph was studied in [8,9] and discussed in [13,18]. A subset $S \subseteq V(G)$ is said to be a vertex covering set of G if every edge has at least one end vertex in S. A vertex covering set of G with the minimum cardinality is called a minimum vertex covering set of G. The vertex covering number of G is the cardinality of any minimum vertex covering set of G. It is denoted by $\alpha(G)$ [19]. A set of vertices of G is said to be monophonic domination set if it is both a monophonic set and a dominating set of G. The minimum cardinality of a monophonic domination set of G is called a monophonic domination number of G and denoted by $\gamma_m(G)$. The monophonic domination number was studied in [14].

The following theorems will be used in the sequel.

Theorem 1.1.[18] Every extreme vertex of a connected graph G belongs to every monophonic set of G. In particular, each end vertex of G belongs to every monophonic set of G.

Theorem1.2.[18] For any tree T with k end vertices, $m(T)=k$. In fact, the set of all end vertices of T is the unique monophonic set of T.

Throughout this paper G denotes a connected graph with at least two vertices.

2. MONOPHONIC VERTEX COVER

Definition2.1. Let G be a connected graph of order $n \geq 2$. A set S of vertices of G is a monophonic vertex cover of G if S is both a monophonic set and a vertex cover of G. The minimum cardinality of a monophonic vertex cover of G is called the monophonic vertex covering number of G and is denoted by m_α(G). Any monophonic vertex cover of cardinality m_α (G) is a m_α-set of G.

Example2.2. For the graph G given in Figure 2.1, S = $\{v_1, v_5\}$ is a minimum monophonic set of G so that m(G) = 2 and $S' = \{v_1, v_4, v_5\}$ is a minimum monophonic vertex cover of G so that m_α (G)=3. Thus the monophonic number is different from the monophonic vertex covering number of a graph G.

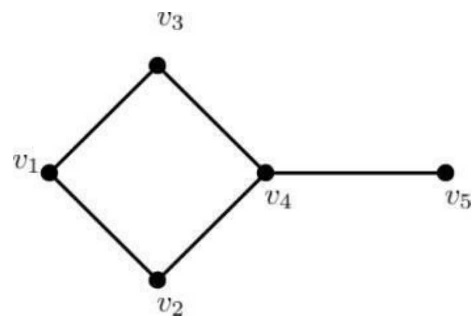

Figure 2.1 G

Remark 2.3. For the graph G given in Figure 2.2, $S = \{v_2, v_3\}$ is a minimum monophonic set of G so that $m(G) = 2$. S is also a minimum monophonic dominating set of G so that γ_m(G)=2. $S' = \{v_1, v_2, v_3\}$ is a minimum monophonic vertex cover of G so that m_α (G)=3. Hence the monophonic vertex covering number of a graph is different from the monophonic number and monophonic dominating number of a graph G.

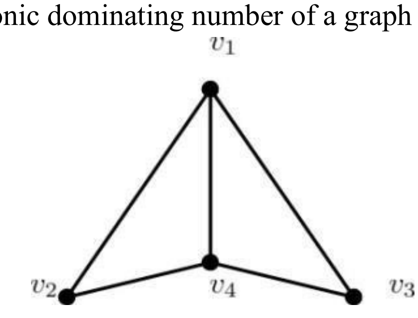

Figure 2.2 G

Theorem 2.4. For any connected graph G, $2 \leq max\ \{\alpha(G), m(G)\} \leq m_\alpha\ (G) \leq n$.

Proof of theorem 2.4. Any monophonic set of G needs at least 2 vertices. Then $2 \leq max\{\alpha(G), m(G)\}$. From the definition of monophonic vertex cover of G, we have, $max\ \{\alpha(G), m(G)\} \leq m_\alpha\ (G)$. Clearly V($G$) is a monophonic vertex cover of G. Hence $m_\alpha\ (G) \leq n$. Thus $2 \leq max\ \{\alpha(G), m\ (G)\} \leq m_\alpha\ (G) \leq n$.

Remark 2.5. The bounds in Theorem 2.4 are sharp. For the complete graph K_4, m_α (K_4) = 4. The bounds are strict in Figure 2.3 as $\alpha(G)$=2*,* $m(G)$=3, m_α (G)=4. Here 2<3<4<5.

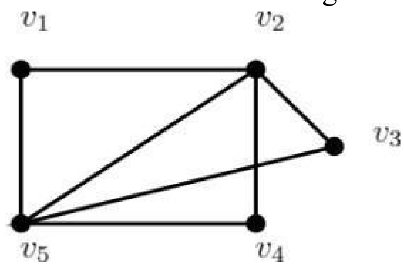

Figure 2.3 G

Remark 2.6. Clearly union of a vertex covering set and a monophonic set of G is a monophonic vertex cover of G. In Figure 2.1, $S = \{v_1, v_4, v_5\}$ is a monophonic vertex cover and in Figure 2.2, S= $\{v_1, v_2, v_3, v_4\}$ is a monophonic vertex cover.

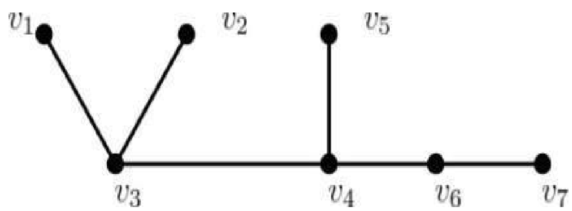

Figure 2.4 G

Thus 2 ≤ *max* $\{\alpha(G), m(G)\} \leq m_\alpha$ $(G) \leq$ *min* $\{\alpha(G)+m(G), n\}$.

For the graph G in Figure 2.4, we observe that S_1= $\{v_3, v_5, v_6\}$ is a minimum vertex cover of G so that $\alpha(G)$ = 3, S_2 = $\{v_1, v_2, v_5, v_7\}$ is a minimum monophonic set of G so that m(G)=4 and S_3= $\{v_1, v_2, v_3, v_5, v_6, v_7\}$ = $S_1 \cup S_2$ is a m_α-set of G and so m_α (G) = 6 < n =7.

Theorem 2.7. Each extreme vertex of G belongs to every monophonic vertex cover of G. In particular, each end vertex of G belongs to every monophonic vertex cover of G.

Proof of theorem 2.7. From the definition of m_α-set, every m_α-set of G is a m-set of G. Hence the result follows from Theorem 1.1.

Corollary2.8. For any graph G with k extreme vertices, $max\{2, k\} \leq m_\alpha$ $(G) \leq n$.

Proof of corollary 2.8. The result follows from Theorem 2.4 and Theorem 2.7.

Corollary 2.9. Let $K_{1,n-1}$(n≥3) be a star. Then m_α $(K_{1,n-1})$ = n −1.

Proof of corollary 2.9. Let x be the centre and S = $\{v_1, v_2, \ldots, v_{n-1}\}$ be the set of all extreme vertices of $K_{1,n-1}$(n ≥ 3). Clearly S is a minimum monophonic vertex cover of $K_{1,n-1}$(n≥3) by Theorem 2.7. Hence m_α $(K_{1,n-1})$ = n−1.

Corollary2.10. For the complete graph K_n(n≥2), m_α (K_n)=n.

Proof of corollary 2.10. We have every vertex of the complete graph K_n(n≥2) is an extreme vertex. Then by Theorem 2.7, the vertex set is the unique monophonic vertex cover of K_n. Then $m_\alpha(K_n)$ = n.

Theorem2.11. If G is a connected graph of order $n \geq 2$, then

(i) m_α (G) = 2 if and only if G is either K_2 or $K_{2,n-2}$(n≥3).

(ii) m_α (G) = n if and only if G = K_n(n≥2).

Proof of theorem 2.11.

(i) Let m_α (G)=2. Let S = {u,v} be a minimum monophonic vertex cover of G. We claim that G = K_2 or $K_{2,n-2}$ ($n \geq 3$). Suppose that G = K_2. Then there is nothing to prove. If not, then $n \geq 3$ and since $S = \{u, v\}$ is a m_α -set of G, u and v cannot be adjacent in G. Let $W = V - S$. We claim that every vertex of W is adjacent to both u and v and no two vertices of W are adjacent. Suppose there is a vertex $w \in W$ such that w is adjacent to at most one vertex in S. Then w lies on a u-v monophonic path of length at least 3. Let P: $u = v_0, v_1, v_2, \ldots, v_i = w, v_{i+1}, \ldots, v_m = v$ be a u-v monophonic. Then the edges in E(P)- $\{v_0v_1, v_{m-1}v_m\}$ are not covered by any of the vertices u and v, which is a contradiction to S is a m_α -set. Hence every vertex of W is adjacent to both u and v. Suppose there exist vertices $w_i, w_j \in W$ such that w_i and w_j are adjacent. Since every vertex of W is adjacent to both u and v and $S = \{u,v\}$ is a m_α- set of G, w_i and w_j lie on the u-v monophonic paths $uw_i v$ and $uw_j v$ respectively. Then the edge w_iw_j is not covered by any of vertices of S, which is a contradiction to S is a m_α - set of G. Hence no two vertices of W are adjacent in G. Thus G is the complete bi partite graph $K_{2,n-2}$ (n≥3) with the partite sets S and W.
Conversely assume that G = K_2 or $K_{2,n-2}$ ($n \geq 3$). If G = K_2, then by Corollary 2.10, m_α (K_2) = 2. If not, let G = $K_{2,n-2}$ ($n \geq 3$). Let U = $\{u_1u_2\}$ and W = $\{w_1, w_2, \ldots, w_{n-2}\}$ be the bipartition of G. Clearly every vertex w_i(1≤i≤n−2) lies on the monophonic path $u_1w_iu_2$ and the vertices u_1 and u_2 cover all the edges of G. Hence U is a monophonic vertex cover of G and so m_α (G)=2.

(ii) Assume that G = K_n (n≥2). Then by Corollary 2.10, m_α (G) = n. Conversely assume that $m_\alpha(G) = n$. We claim that G = K_n (n≥2). For n=2, the result holds from (i). Let $n \geq 3$. Suppose there exist two non-adjacent vertices u and v in G. Let a vertex x be adjacent to u lying on a u-v monophonic. Then $V(G)-\{x\}$ is a monophonic vertex cover of G, which is a contradiction to $m_\alpha(G) = n$. Thus G=K_n.

Theorem.2.12. For a connected graph G with $m(G) \geq n-1$, m_α (G) = $m(G)$.

Proof of theorem 2.12. Let G be a connected graph with $m(G) \geq n - 1$. Then by Theorem 2.4, $m(G) \leq m_\alpha(G) \leq n$. Now, if $m(G) = n$, then $m_\alpha(G) = n$. Hence m_α (G) = $m(G)$. If $m(G) = n-$

1, then let S = $\{x_1, x_2, \dots, x_{n-1}\}$ be a minimum monophonic set of G. Let $x \notin S$ be a vertex of G. Then any edge xx_i ($1 \leq i \leq n-1$) lies on a monophonic path joining pair of vertices of S and every edge of G has at least one end point in S. Hence S is a minimum monophonic vertex cover of G and so $m_\alpha(G) = m(G)$.

Remark.2.13. The converse of Theorem 2.12 need not be true. For the graph in Figure 2.5, $S = \{v_1, v_2\}$ is both a m-set of G and a m_α-set of G. Hence $m_\alpha(G) = m(G) = 2$ but $m(G) < n-1$.

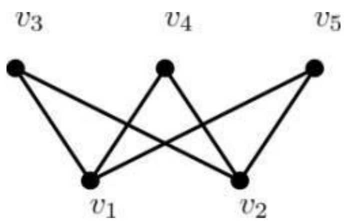

Figure 2.5 G

Theorem.2.14. For a connected graph G of order $n \geq 2$, $m_\alpha(G) = m(G)$ if and only if there exists a minimum monophonic set of G such that $V(G) - S$ is either empty or an independent set.

Proof of theorem 2.14. Assume that $m_\alpha(G) = m(G)$. Let S = $\{v_1, v_2, \dots, v_k\}$ be a minimum monophonic vertex cover of G. Then S is also a minimum monophonic set of G. If n=k, then V(G)– S is empty. Let $n > k$. If not, there exist two vertices $u, v \in V(G) - S$ such that $uv \in E(G)$. Then the edge uv has none of its end vertices in S, which is a contradiction. Hence there exists a minimum monophonic set of G such that V(G)-S is either empty or an independent set. Conversely assume that there exists a minimum monophonic set of G such that V(G)– S is either empty or an independent set. Let S = $\{v_1, v_2, \dots, v_k\}$ so that m(G)=|S|. Suppose V(G)– S is empty. Then n=k and S=V(G). Hence S is a minimum monophonic vertex cover of G so that m_α(G) = m(G). If not, let V(G)– S be independent. Then every edge of G has at least one end in V(G)−(V(G)−S) = S and so S is a vertex cover of G. Thus S is a minimum monophonic vertex cover of G. Thus m_α(G) = m(G).

Theorem.2.15. For the cycle $C_n (n \geq 4)$, $m_\alpha(C_n) = \left\lceil \frac{n}{2} \right\rceil$.

Proof of theorem 2.15. Let C_n: $v_1\ v_2\ \dots\ v_n\ v_1$be a cycle of order n. Here $S = \{v_1, v_3, v_5, \dots, v_{2\left\lceil \frac{n}{2} \right\rceil - 1}\}$ is a minimum monophonic vertex cover of C_n. Hence $m_\alpha(C_n) = \left\lceil \frac{n}{2} \right\rceil$.

Theorem.2.16. Let T be a tree of order $n \geq 2$. Then the following statements are equivalent.

(1) $m_\alpha(T) = m(T)$.

(2) T is a star.

(3) $\alpha(T) = 1$.

(4) The set of all end vertices of T is a vertex cover of T.

Proof of theorem 2.16. Let S be the set of all end vertices of T. Since T is a tree, from the Theorem 1. 2, we have, S is the unique m-set of T.
(1) ⇒ (2) Assume that m_α $(T) = m(T)$.We claim that T is a star. If not, then $diam\ T \geq 3$. Then T has at least one edge other than the end edges. Let S' be the set of all edges of T which are not end edges. Then clearly no edges of S' have its end vertices in S. Hence S is not a vertex cover of T. By Theorem 2.7, any monophonic vertex cover of T contains S. Hence m_α $(T) >$ $|S| = m(T)$, which is a contradiction to $m_\alpha(T)=m(T)$.
(2) ⇒(3) Assume that T is a star. If n=2, then an end vertex of T will cover the edge of T. If $n \geq 3$, then the cut vertex of T will cover all the edges in T. Hence $\alpha(T)=1$.
(3) ⇒(4)Assume that $\alpha(T)=1$.Then there exists a vertex say x in T such that x is an end vertex of all the edges in T. Hence all the edges in T are the end edges in T and so S forms a vertex cover of T.
(4)⇒(1) Assume that S is a vertex cover of T. Then by Theorem 1.2, S is a m-set of T and by Theorem 2.7, S is a m_α-set of T. Hence m_α $(T)=m(T)$.

Remark 2.17. The results in Theorem 2.16 are not equivalent for any connected graph G of order $n \geq 2$. For the graph G in Figure 2.6, S= $\{v_1, v_2, v_3\}$ is both m-set andm_α-set of G. So $m_\alpha(G) = m(G) = 3$. Also, S is a minimum vertex covering set and so $\alpha(G)$=3. And here G is not a star.

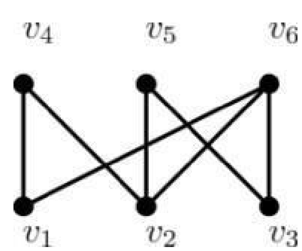

Figure 2.6 G

3. Conclusion

In this paper we analyzed the monophonic vertex covering number of a graph. It is more interesting to continue my research in this area and it is very useful for further research.

REFERENCES

1. A.M. Anto, P. Paul Hawkins, Vertex Polynomial of Graphs with New Results, Global Journal of Pure and Applied Mathematics, Volume 15, Issue 4, Pages 469-475 (2019).

2. A. M. Anto, P. Paul Hawkins, T. Shyla Isac Mary, Perfect Dominating Sets and Perfect Dominating Polynomial of a Cycle, Advances in Mathematics: Scientific Journal, Volume 8, Issue 3, Pages 538-543 (2019).

3. A. M. Anto, P. Paul Hawkins, T. Shyla Isac Mary, Perfect Dominating Sets and Perfect Dominating Polynomial of a Path, International Journal of Advanced Science and Technology, Volume 28, Issue 16, Pages 1226-1236 (2019).

4. F.Buckley and F. Harary, Distance in Graphs, Addison - Wesley, Redwood City, (1990).

5. Buckley, Fand Harary, F and Quintas, L V, Extremal results on the geodetic number of a graph, ScientiaA,2(1), (1988).

6. G. Chartrand, F. Harary and P.Chang, On the Geodetic Number of a Graph, Networks: An International Journal,39(1),16(2002).

7. Chartrand, Garyand Johns, Garry Land Zhang, Ping, On the detour number and geodetic number of a graph, Ars Combinatoria, vol72,3–15, (2004).

8. S. Durai raj, K. A. Francis Jude Shini, X. Lenin Xaviour, and Anto A. M. On The Study Of Edge Monophonic Vertex Covering Number. *Ratio Mathematica,* vol 44, Pages 197-204 (2022).

9. S. Durai raj, K. A. Francis Jude Shini, X. Lenin Xaviour, and Anto A. M. The Monophonic Vertex Covering Number of Power of Cycles. *Journal of Data Acquisition and Processing,* Vol.38, Issue 3, Pages 1115-1120 (2023).

10. S. Durai Raj, S. G. Shiji Kumari and A. M. Anto, Certified Domination Number in Corona Product of Graphs, Malaya Journal of Matematik, Vol. 9, No.1, Page No. 1080-1082 (2021).

11. S. Durai Raj, S. G. Shiji Kumari and A. M. Anto, Certified Domination Number in Subdivision of Graphs, International Journal of Mechanical Engineering, Vol. 6, Issue 3, Page No. 4324-4328.

12. Harary, Frank, Graph Theory, Addison Wesley Publishing Company (1969).

13. John, J and Panchali, S, The upper monophonic number of a graph, International J.Math. Combin, vol4,46–52(2010).

14. John, J and Sudhahar, P Arul Paul, The monophonic domination number of a graph, Proceedings of the International Conference on Mathematics and Business Management, vol1,142– 145, (2012).

15. X.Lenin Xaviour and S.Robinson Chellathurai, Geodetic Global Domination in Corona and Strong Product of Graphs, Discrete Mathematics, Algorithms and Applications,vol12(4) doi.org/10.1142/S1793830920500433(2020).

16. X. Lenin Xaviour and S. Robinson Chellathurai, On the Upper Geodetic Global Domination Number of a Graph, Proyecciones Journal of Mathematics, vol39(6) (2020).

17. S.Robinson Chellathurai and X.Lenin Xaviour, Geodetic Global Domination in Graphs, International Journal of Mathematical Archive, 29-36(2018).

18. Santhakumaran, AP and Titus, P and Ganesamoorthy, K, On the monophonic number of a graph, Journal of applied mathematics and informatics, Vol 32(2), 255–266, (2014).

19. Thakkar, DK and Bosamiya, JC, Vertex covering number of a graph, Mathematics Today, Vol27,30-30(2011).