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A MIXTURE WEIBULL RAYLEIGH DISTRIBUTION FOR FITTING FAILURE TIMES DATA

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Abstract: Modeling and analysis of lifetime data is an important aspect of statistical work in a wide variety of applications. However, the data in many applications such as economics, engineering, biological studies, environmental sciences, medical sciences and finance can be considered as data coming from a mixture population of two or more different distributions. Mixture classical distributions have a limited ability to represent real data. A new mixture model called Weibull Rayleigh mixture model is introduced in this paper. The proposed model is based on components of the composite Weibull Rayleigh distribution. The interest of this model is that the density function can take different possible shapes, symmetric and asymmetric. Moreover, the behaviour of the related hazard function varies and can increase or decrease. The new model turns out to be quite flexible for modeling positive data. The maximum likelihood estimation method is applied to obtain the estimators of the parameters of the new model based on Type-I, Type-II censored samples and complete samples. The statistical characteristics of this distribution are obtained such as moment, incomplete moment, order statistic and others. A Monte Carlo simulation study is employed to check the consistency of the estimates of model parameters using different sample sizes. The model's performance is evaluated by comparing it to other competing distributions using three sets of real data. The proposed model is superior to its counterparts' models in representing the different data sets.
Keywords Weibull Rayleigh distribution, maximum likelihood estimation bias, means squared error, Type-I censored samples, Type-II censored samples.

1. Introduction

Lifetime data are present widely in many different applications. However, the data in applications such as economics, engineering, biological studies, medical and environmental sciences can be considered as data coming from mixture population of two or more different distributions. Most of researchers focus on extending and modifying the existing classical distribution in order to provide flexibility in modeling data from different mixture populations greater. Recently, mixture modeling has become widely used to model data from different mixture populations. The main aim of this paper is a generating new model by applying the mixture modeling with Weibull Rayleigh distribution.

The using of finite mixture models is very old in the history of statistics. The first use of finite mixture models was in the nineteenth century in a paper by Newcomb [1] who used it in the context of modeling outliers. Pearson [2] studied a mixture of two univariate Gaussian distribution and employed the method of moments for estimating the model parameters. He used the mixture approach to analyze a data set containing ratios of forehead to body lengths for 1,000 crabs. Figueiredo and Jain [3] used the finite mixture to unsupervised learning models. Franco et al. [4] discussed the classification of the aging properties of generalized mixtures of two or three Weibull distributions in terms of the mixing weights, scale parameters and a common shape parameter. Razali and Al-Wakeel [5] used the mixture of two and three Weibull distributions to analyze the data of failure times. Zhang et al. [6] introduced a mixture Weibull proportional hazards model to predict the failure of a mechanical system with multiple failure modes. Qutb et al. [7] studied a mixture of two Weibull distributions with a common shape parameter, based on the generalized order statistics. Huang et al. [8] discussed likelihood method for finite multivariate Gaussian mixture models. Zong et al. [9] studied a deep autoencoding Gaussian mixture model for unsupervised anomaly. McLachlan et al. [10] provided the methodological and theory for the applications of finite mixture models and discussed the role of mixture models in clustering of independent and identically distributed data. They also applied the maximum likelihood (ML) estimation and the moment estimation methods for parametric mixture models. Teamah et al. [11] introduced a new mixture distribution as a result of mixing Fr´echet-Weibull distribution with exponential distribution; it is called Fr´echet-Weibull mixture exponential distribution and used the ML estimation for estimating the parameters of the mixture distribution. In other hand, many approaches have been proposed to generate new families of models by employing one or more additional shape parameter(s) to the baseline distribution. The benefit of this induction of parameter(s) is in exploring tail properties and improving the goodness-of-fit of the generator family. Bourguignon et al. [12] introduced a new generator based on the Weibull random variable called the new Weibull-G family by introducing one or more shape parameter to the baseline distribution for defining new generators for continuous families of distributions.

In this paper we will introduce a mixture of two component of Weibull Rayleigh distribution. The statistical properties of the proposed mixture distribution are obtained. Also, we will provide a comparison of ML estimators of the model parameters based on complete samples, Type-I and Type-II censored samples through Monte Carlo simulations. The bias and the mean squared error (MSE) are used as measures for comparison. Moreover, the usefulness of the proposed distribution is investigated through three numerical applications.

2. Two-Components Mixture Weibull Rayleigh Distribution

2.1. Definition of Mixture Distribution

The probability density function (PDF) of the Weibull Rayleigh distribution is given as

$$f(x) = 2\,\mathrm{c}\left(\frac{\lambda}{\beta}\right)^{c} x^{2\mathrm{c}-1} e^{-\left(\frac{\lambda x^2}{\beta}\right)^{c}}, x \geq 0, c, \lambda, \beta > 0, \tag{1}$$

and the cumulative distribution function (CDF) is given as

$$F(x) = 1 - e^{-\left(\frac{\lambda x^2}{\beta}\right)^{c}}. \tag{2}$$

The hazard rate function (HZF) is given as

$$h(x) = 2\,c\,\left(\frac{\lambda}{\beta}\right)^{c} x^{2c-1}. \tag{3}$$

2.2. Mixture of a Two Weibull Rayleigh Distribution

A density function for the mixture of two components densities with mixing proportions p and $1-p$ is defined as

$f(x) = pf_1(x) + (1-p)\,f_2(x)$

$$f(x) = 2\,p\,c_1\left(\frac{\lambda_1}{\beta_1}\right)^{c_1} x^{2c_1-1}e^{-\left(\frac{\lambda_1 x^2}{\beta_1}\right)^{c_1}} + 2(1-p)\,c_2\left(\frac{\lambda_2}{\beta_2}\right)^{c_2} x^{2c_2-1}\,e^{-\left(\frac{\lambda_2 x^2}{\beta_2}\right)^{c_2}}. \tag{4}$$

where p satisfies the condition, $0 \le p \le 1$.The CDF for the mixture model is defined as

$F(x) = pF_1(x) + (1-p)F_2(x),$

$$F(x) = p\left(1 - e^{-\left(\frac{\lambda_1 x^2}{\beta_1}\right)^{c_1}}\right) + (1-p)\left(1 - e^{-\left(\frac{\lambda_2 x^2}{\beta_2}\right)^{c_2}}\right). \tag{5}$$

The reliability function (RF) for the mixture model is given as

$R(x) = pR_1(x) + (1-p)R_2(x),$

$$R(x) = p\,e^{-\left(\frac{\lambda_1 x^2}{\beta_1}\right)^{c_1}} + (1-p)\,e^{-\left(\frac{\lambda_2 x^2}{\beta_2}\right)^{c_2}}. \tag{6}$$

The HZF is given as

$$h(x) = 2pc_1\left(\frac{\lambda_1}{\beta_1}\right)^{c_1} x^{2c_1-1} + 2(1-p)c_2\left(\frac{\lambda_2}{\beta_2}\right)^{c_2} x^{2c_2-1}. \tag{7}$$

The reversed HZF is given as

$$rh(x) = p\,\frac{2\;c_1\left(\frac{\lambda_1}{\beta_1}\right)^{c_1} x^{2c_1-1}e^{-\left(\frac{\lambda_1 x^2}{\beta_1}\right)^{c_1}}}{1 - e^{-\left(\frac{\lambda_1 x^2}{\beta_1}\right)^{c_1}}} + (1-p)\,\frac{2\,c_2\left(\frac{\lambda_2}{\beta_2}\right)^{c_2} x^{2c_2-1}\,e^{-\left(\frac{\lambda_2 x^2}{\beta_2}\right)^{c_2}}}{1 - e^{-\left(\frac{\lambda_2 x^2}{\beta_2}\right)^{c_2}}}. \tag{8}$$

The quantile function of X with CDF in (5) is given by

$$Q(u) = p\left[\frac{\beta_1}{\lambda_1}\big(Log(1-u)\big)^{\frac{1}{c_1}}\right]^{\frac{1}{2}} + (1-p)\left[\frac{\beta_2}{\lambda_2}\big(Log(1-u)\big)^{\frac{1}{c_2}}\right]^{\frac{1}{2}}. \tag{9}$$

2.3. Graphical description

The plots of PDF and the HZF are displayed for different values of parameters in Figure1. The figure shows several forms for PDF and HZF curve. This indicates that the new model is flexible and may be suitable for different types of data.

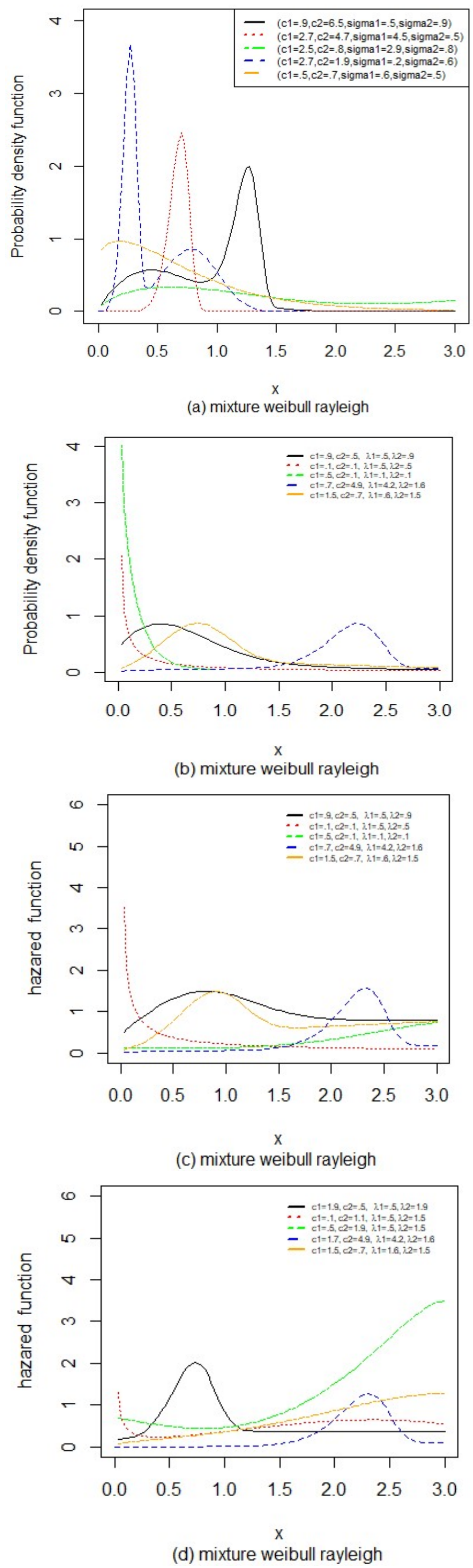

Figure 1. Plots of PDF and HZF of the (MWR) distribution.

3. General Properties

Moment

The finite mixture of the r^{th} moments of the two components is given as

$$\acute{\mu}_r = \sum_{j=1}^{2} p_j\, \mu_{il} = 2\int p\, c_1 \left(\frac{\lambda_1}{\beta_1}\right)^{c_1} x^{r+2c_1-1} e^{-\left(\frac{\lambda_1 x^2}{\beta_1}\right)^{c_1}} dx$$
$$+2\int (1-p)\, c_2 \left(\frac{\lambda_2}{\beta_2}\right)^{c_2} x^{r+2c_2-1} e^{-\left(\frac{\lambda_2 x^2}{\beta_2}\right)^{c_2}} dx$$
$$E(x^r) = p\left(\frac{\beta_1}{\lambda_1}\right)^{\frac{r}{2}} \sqrt{\left(\frac{r}{2c_1}+1\right)} + (1-p)\left(\frac{\beta_2}{\lambda_2}\right)^{\frac{r}{2}} \sqrt{\left(\frac{r}{2c_2}+1\right)}. \tag{10}$$

The mean is given when $r = 1$ as follows

$$\acute{\mu}_1 = p\left(\frac{\beta_1}{\lambda_1}\right)^{\frac{1}{2}} \sqrt{\left(\frac{1}{2c_1}+1\right)} + (1-p)\left(\frac{\beta_2}{\lambda_2}\right)^{\frac{1}{2}} \sqrt{\left(\frac{1}{2c_2}+1\right)}. \tag{11}$$

The variance is defined as follows

$$\sigma^2 = E(x^2) - \acute{\mu}_1{}^2. \tag{12}$$

The moments generating function of MWR distribution is represented as

$$M_X(t) = p[M_{X1}(t)] + (1-p)[M_{X2}(t)].$$

$$M_X(t) = \int e^{tx} f_j(x)dx = \int e^{tx}\left\{p\left[2\, c_1 \left(\frac{\lambda_1}{\beta_1}\right)^{c_1} x^{2c_1-1}\, e^{-\left(\frac{\lambda_1 x^2}{\beta_1}\right)^{c_1}}\right]\right. \tag{13}$$
$$\left. +(1-p)\left[2\, c_2 \left(\frac{\lambda_2}{\beta_2}\right)^{c_2} x^{2c_2-1}\, e^{-\left(\frac{\lambda_2 x^2}{\beta_2}\right)^{c_2}}\right]\right\} dx.$$

So, the moments generating function of MWR distribution can be written as

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\left\{p\left(\frac{\beta_1}{\lambda_1}\right)^{\frac{r}{2}} \sqrt{\left(\frac{r}{2c_1}+1\right)} + (1-p)\left(\frac{\beta_2}{\lambda_2}\right)^{\frac{r}{2}} \sqrt{\left(\frac{r}{2c_2}+1\right)}\right\}. \tag{14}$$

Incomplete Moments

The r^{th}incomplete moment of MWR distribution is given as

$$T_r(z) = \sum_{j=1}^{2} p_j \mu_{jr} = \int^{z} 2\, p\, c_1 \left(\frac{\lambda_1}{\beta_1}\right)^{c_1} x^{2c_1-1} e^{-\left(\frac{\lambda_1 x^2}{\beta_1}\right)^{c_1}} x^r\, dx \tag{15}$$
$$+\int^{z} 2(1-p)\, c_2 \left(\frac{\lambda_2}{\beta_2}\right)^{c_2} x^{2c_2-1}\, e^{-\left(\frac{\lambda_2 x^2}{\beta_2}\right)^{c_2}} x^r\, dx.$$

The first incomplete moment of a finite mixture of k components equals

$$T_1(z) = \sum_{j=1}^{2} p_j \mu_{j1} = \int^{z} 2\, p\, c_1 \left(\frac{\lambda_1}{\beta_1}\right)^{c_1} x^{2c_1-1} e^{-\left(\frac{\lambda_1 x^2}{\beta_1}\right)^{c_1}} x\, dx + \int^{z} 2(1-p)\, c_2 \left(\frac{\lambda_2}{\beta_2}\right)^{c_2} x^{2c_2-1} e^{-\left(\frac{\lambda_2 x^2}{\beta_2}\right)^{c_2}} x\, dx. \quad (16)$$

Mean Deviations

The mean deviation about the mean μ of MWR distribution is given as

$$\delta 1 = \int_D |x-\mu| \left[p\left(2\, p\, c_1 \left(\frac{\lambda_1}{\beta_1}\right)^{c_1} x^{2c_1-1} e^{-\left(\frac{\lambda_1 x^2}{\beta_1}\right)^{c_1}} \right) + (1-p)\left(2(1-p)\, c_2 \left(\frac{\lambda_2}{\beta_2}\right)^{c_2} x^{2c_2-1} e^{-\left(\frac{\lambda_2 x^2}{\beta_2}\right)^{c_2}} \right) \right] dx.$$

$$\delta 1 = p\left(2\, p\, c_1 \left(\frac{\lambda_1}{\beta_1}\right)^{c_1} x^{2c_1-1} e^{-\left(\frac{\lambda_1 x^2}{\beta_1}\right)^{c_1}} \right) + (1-p)\left(2\, c_2 \left(\frac{\lambda_2}{\beta_2}\right)^{c_2} x^{2c_2-1} e^{-\left(\frac{\lambda_2 x^2}{\beta_2}\right)^{c_2}} \right). \quad (17)$$

and the mean deviation about the median M equals

$$\delta 2 = \int_D |x-M| \left[p\left(2\, p\, c_1 \left(\frac{\lambda_1}{\beta_1}\right)^{c_1} x^{2c_1-1} e^{-\left(\frac{\lambda_1 x^2}{\beta_1}\right)^{c_1}} \right) + (1-p)\left(2(1-p)\, c_2 \left(\frac{\lambda_2}{\beta_2}\right)^{c_2} x^{2c_2-1} e^{-\left(\frac{\lambda_2 x^2}{\beta_2}\right)^{c_2}} \right) \right] dx. \quad (18)$$

Since, the median is given as $F(x;\lambda,c,\beta) = \frac{1}{2}$, these forms can be written as

$$\delta 1 = 2\mu\, F(x;\lambda,c,\beta) - 2T_1(\mu), \quad \text{and} \quad \delta 2 = \mu - 2T_1(M), \quad (19)$$

where $T_1(z)$ is the first incomplete moment of X obtained from (16). Therefore

$$\delta 1_j = 2\mu_j F_j\left(x;\lambda_j,c_j,\beta_j\right) - 2T_1(\mu_j), \quad \text{and} \quad \delta 2_j = \mu_j - 2T_1(M_j), \quad (20)$$

where

$$\delta 1 = 2\mu\left(p\left[1 - e^{-\left(\frac{\lambda_1 x^2}{\beta_1}\right)^{c_1}}\right] + (1-p)\left[1 - e^{-\left(\frac{\lambda_2 x^2}{\beta_2}\right)^{c_2}}\right] \right) - 2T_1(\mu), \quad (21)$$

and

$$\delta 2 = \mu - 2T_1(M).$$

Rényi Entropy

The Rényi entropy of MWR model is expressed as

$$H_R^S(x) = \frac{1}{1-S} log\left(\int_D \left[\left(2\, p\, c_1 \left(\frac{\lambda_1}{\beta_1}\right)^{c_1} x^{2c_1-1} e^{-\left(\frac{\lambda_1 x^2}{\beta_1}\right)^{c_1}}\right) + (1-p)\left(2(1-p)\, c_2 \left(\frac{\lambda_2}{\beta_2}\right)^{c_2} x^{2c_2-1} e^{-\left(\frac{\lambda_2 x^2}{\beta_2}\right)^{c_2}}\right)\right]^S dx\right). \quad (22)$$

It is a difficult problem to obtain $H_R^S(x)$ in closed-form for the mixture model.

Shannon entropy

The Shannon entropy of X is represented as

$$H_s(x) = E_X\{-log(f(x;\lambda, c, \beta))\}, \quad (23)$$

where the log-likelihood function is given as

$$log[f_j(x; c_j, \lambda_j, \beta_j)] = log\ 2 + log\ p + log\, c_1 + c_1\, log\, \lambda_1 - c_1 log\, \beta_1 + (2c_1 - 1)\, log\, x - \left(\frac{\lambda_1 x^2}{\beta_1}\right)^{c_1} + log\ 2 + log\, (1-p) + log\, c_2 + c_2 log\, \lambda_2 - c_2\, log\, \beta_2 + (2c_2 - 1)\, log\, x - \left(\frac{\lambda_2 x^2}{\beta_2}\right)^{c_2}. \quad (24)$$

Thus, it can be reduced to

$$H_s(x) = -log\ 2 - log\ p - log\, c_1 - c_1\, log\ \lambda_1 + c_1 log\, \beta_1 - (2c_1 - 1)\, E(log\, X) + E\left(\frac{\lambda_1 X^2}{\beta_1}\right)^{c_1} - log\ 2 - log\, (1-p) - log\, c_2 - c_2 log\, \lambda_2 + c_2\, log\, \beta_2 - (2c_2 - 1) E(log\, X) + E\left(\frac{\lambda_2 X^2}{\beta_2}\right)^{c_2}. \quad (25)$$

Distribution of order statistic

The r^{th} order statistics for the MWR distribution can be written as

$$f_{r,n}(x) = \frac{n!}{(n-r)!\,(r-1)!} \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} \times \left(2\, p\, c_1 \left(\frac{\lambda_1}{\beta_1}\right)^{c_1} x^{2c_1-1} e^{-\left(\frac{\lambda_1 x^2}{\beta_1}\right)^{c_1}} + 2(1-p)\, c_2 \left(\frac{\lambda_2}{\beta_2}\right)^{c_2} x^{2c_2-1} e^{-\left(\frac{\lambda_2 x^2}{\beta_2}\right)^{c_2}}\right) \quad (26)$$

$$\times\left[p\left(1-e^{-\left(\frac{\lambda_1 x^2}{\beta_1}\right)^{c_1}}\right)+(1-p)\left(1-e^{-\left(\frac{\lambda_2 x^2}{\beta_2}\right)^{c_2}}\right)\right]^{r+i-1}.$$

Special Cases:

- If $r=1$, in (26), the PDF of the smallest order statistic can be obtained.
- If $r=n$, in (26), the PDF of the largest order statistic can be obtained.
- If $r=\frac{n+1}{2}$, in (26), the PDF of the median observable in the odd sample size case can be obtained.

4. Estimation of Mixture Weibull Rayleigh Distribution

In this section, the parameters of MWR distribution is estimated by maximum likelihood estimation method with complete sample and censoring samples of Type-I and Type

4.1. Maximum likelihood estimation based on complete sample

If $x_1, x_2, \dots, x_n$ is a random sample of size n from the MWR distribution, then the log likelihood function for the vector of parameters $\theta_j = (c_j, \lambda_j, \beta_j)$ is given as

$$\ell(c_j,\lambda_j,\beta_j)=\sum_{i=1}^{n}\left[l\,og\left(\sum_{j=1}^{k}2\,p_j\,c_j\left(\frac{\lambda_j}{\beta_j}\right)^{c_j}x_i^{2c_j-1}\,e^{-\left(\frac{\lambda_j x_i^2}{\beta_j}\right)^{c_j}}\right)\right]. \tag{27}$$

The ML estimators can be computed by differentiating (27) with respect to each parameter as follows

$$\frac{\partial\ell}{\partial c_j}=\sum_{i=1}^{n}\left[\frac{2p_j\left(\frac{\lambda_j}{\beta_j}\right)^{c_j}x_i^{2c_j-1}e^{-\left(\frac{\lambda_j x_i^2}{\beta_j}\right)^{c_j}}\left[1+c_j\left(ln\left(\frac{\lambda_j}{\beta_j}\right)+2\ln(x_i)-\left(\frac{\lambda_j x_i^2}{\beta_j}\right)^{c_j}ln\left(\frac{\lambda_j x_i^2}{\beta_j}\right)\right)\right]}{\sum_{j=1}^{k}2\,p_j c_j\left(\frac{\lambda_j}{\beta_j}\right)^{c_j}x_i^{2c_j-1}\,e^{-\left(\frac{\lambda_j x_i^2}{\beta_j}\right)^{c_j}}}\right] \tag{28}$$

$$\frac{\partial\ell}{\partial\lambda_j}=\sum_{i=1}^{n}\left[\frac{2p_j\frac{c_j^{\,2}}{\lambda_j}\left(\frac{\lambda_j}{\beta_j}\right)^{c_j}x_i^{2c_j-1}\,e^{-\left(\frac{\lambda_j x_i^2}{\beta_j}\right)^{c_j}}\left[1-\left(\frac{\lambda_j x_i^2}{\beta_j}\right)^{c_j}\right]}{\sum_{j=1}^{k}2\,p_j c_j\left(\frac{\lambda_j}{\beta_j}\right)^{c_j}x_i^{2c_j-1}\,e^{-\left(\frac{\lambda_j x_i^2}{\beta_j}\right)^{c_j}}}\right], \tag{29}$$

$$\frac{\partial \ell}{\partial \beta_j} = -\sum_{i=1}^{n} \left[\frac{2p_j \frac{{c_j}^2}{\beta_j} \left(\frac{\lambda_j}{\beta_j}\right)^{c_j} x_i^{2c_j-1} e^{-\left(\frac{\lambda_j {x_i}^2}{\beta_j}\right)^{c_j}} \left[1 - \left(\frac{\lambda_j {x_i}^2}{\beta_j}\right)^{c_j}\right]}{\sum_{j=1}^{k} 2\, p_j c_j \left(\frac{\lambda_j}{\beta_j}\right)^{c_j} {x_i}^{2c_j-1} e^{-\left(\frac{\lambda_j {x_i}^2}{\beta_j}\right)^{c_j}}} \right], \tag{30}$$

where $j = 1,2$, and $p_1 = p, p_2 = 1 - p$. The MLEs for each parameter can be derived either by solving the system of non-linear equations in (21), (22) and (23) numerically or by maximizing (20) by optimization techniques using the programming language R.

4.2. Maximum likelihood estimation based on Type-I censored samples

If $x_1, x_2, \dots, x_n$ is a random sample of size n from the MWR distribution, then the log likelihood function of Type-I censored sample for the vector of parameters $\theta_j = (c_j, \lambda_j, \beta_j)$ is given as

$$\ell(c_j, \lambda_j, \beta_j) = \sum_{i=1}^{m} \left[\delta_i \, log\left(\sum_{j=1}^{k} 2\, p_j\, c_j \left(\frac{\lambda_j}{\beta_j}\right)^{c_j} {x_i}^{2c_j-1} e^{-\left(\frac{\lambda_j {x_i}^2}{\beta_j}\right)^{c_j}} \right) + (1-\delta_i)\, log\left(\sum_{j=1}^{k} p_j\, e^{-\left(\frac{\lambda_j {x_m}^2}{\beta_j}\right)^{c_j}} \right) \right]. \tag{31}$$

The ML estimators can be computed by differentiating (31) with respect to each parameter as follows

$$\frac{\partial \ell}{\partial c_j} = \sum_{i=1}^{m} \left[\frac{2\delta_i p_j \left(\frac{\lambda_j}{\beta_j}\right)^{c_j} {x_i}^{2c_j-1} e^{-\left(\frac{\lambda_j {x_i}^2}{\beta_j}\right)^{c_j}} \left[1 + c_j \left(ln\left(\frac{\lambda_j}{\beta_j}\right) + 2ln(x_i) - \left(\frac{\lambda_j {x_i}^2}{\beta_j}\right)^{c_j} ln\left(\frac{\lambda_j x}{\beta_j}\right.\right.\right.}{\sum_{j=1}^{k} 2\, p_j c_j \left(\frac{\lambda_j}{\beta_j}\right)^{c_j} {x_i}^{2c_j-1} e^{-\left(\frac{\lambda_j {x_i}^2}{\beta_j}\right)^{c_j}}} - \frac{(1-\delta_i)\, p_j\, e^{-\left(\frac{\lambda_j {x_m}^2}{\beta_j}\right)^{c_j}} \left(\frac{\lambda_j {x_m}^2}{\beta_j}\right)^{c_j} ln\left(\frac{\lambda_j {x_m}^2}{\beta_j}\right)}{\sum_{j=1}^{k} p_j\, e^{-\left(\frac{\lambda_j {x_m}^2}{\beta_j}\right)^{c_j}}} \right], \tag{32}$$

$$\frac{\partial \ell}{\partial \lambda_j} = \tag{33}$$

$$\sum_{i=1}^{m}\left[\frac{2\delta_i p_j \frac{c_j^2}{\lambda_j}\left(\frac{\lambda_j}{\beta_j}\right)^{c_j} x_i^{2c_j-1}\, e^{-\left(\frac{\lambda_j x_i^2}{\beta_j}\right)^{c_j}}\left[1-\left(\frac{\lambda_j x_i^2}{\beta_j}\right)^{c_j}\right]}{\sum_{j=1}^{k} 2\, p_j c_j \left(\frac{\lambda_j}{\beta_j}\right)^{c_j} x_i^{2c_j-1}\, e^{-\left(\frac{\lambda_j x_i^2}{\beta_j}\right)^{c_j}}} - \frac{(1-\delta_i)\, p_j \frac{c_j}{\lambda_j}\left(\frac{\lambda_j x_m^2}{\beta_j}\right)^{c_j} e^{-\left(\frac{\lambda_j x_m^2}{\beta_j}\right)^{c_j}}}{\sum_{j=1}^{k} p_j\, e^{-\left(\frac{\lambda_j x_m^2}{\beta_j}\right)^{c_j}}}\right],$$

$$\frac{\partial \ell}{\partial \beta_j} = \sum_{i=1}^{m}\left[\frac{-2\delta_i\, p_j \frac{c_j^2}{\beta_j}\left(\frac{\lambda_j}{\beta_j}\right)^{c_j} x_i^{2c_j-1}\, e^{-\left(\frac{\lambda_j x_i^2}{\beta_j}\right)^{c_j}}\left[1-\left(\frac{\lambda_j x_i^2}{\beta_j}\right)^{c_j}\right]}{\sum_{j=1}^{k} 2\, p_j c_j \left(\frac{\lambda_j}{\beta_j}\right)^{c_j} x_i^{2c_j-1}\, e^{-\left(\frac{\lambda_j x_i^2}{\beta_j}\right)^{c_j}}} + \frac{(1-\delta_i)\, p_j \frac{c_j}{\beta_j}\left(\frac{\lambda_j x_m^2}{\beta_j}\right)^{c_j} e^{-\left(\frac{\lambda_j x_m^2}{\beta_j}\right)^{c_j}}}{\sum_{j=1}^{k} p_j\, e^{-\left(\frac{\lambda_j x_m^2}{\beta_j}\right)^{c_j}}}\right]. \tag{34}$$

Where $j = 1,2,$ and $p_1 = p, p_2 = 1 - p$. The ML estimators for each parameter can be derived either by solving the system of non-linear equations in (25), (26) and (27) numerically or by maximizing (24) by optimization techniques using the programming language R.

4.3 Maximum likelihood estimation based on Type II censored samples

If $x_1, x_2, \dots, x_n$ is a random sample of size n from the MWR distribution, then the log likelihood function of Type II censored sample for the vector of parameters $\theta_j = (c_j, \lambda_j, \beta_j)$ is given as

$$\ell(c_j, \lambda_j, \beta_j) = log\frac{n!}{(n-m)!} + \sum_{i=1}^{m} log\left(\sum_{j=1}^{k} 2\, p_j\, c_j \left(\frac{\lambda_j}{\beta_j}\right)^{c_j} x_i^{2c_j-1}\, e^{-\left(\frac{\lambda_j x_i^2}{\beta_j}\right)^{c_j}}\right) + (n-m)\, log\left(\sum_{j=1}^{k} p_j\, e^{-\left(\frac{\lambda_j x_m^2}{\beta_j}\right)^{c_j}}\right). \tag{35}$$

The ML estimators can be computed by differentiating (35) with respect to each parameter as follows

$$\frac{\partial \ell}{\partial c_j} = \sum_{i=1}^{m} \frac{2p_j \left(\frac{\lambda_j}{\beta_j}\right)^{c_j} x_i^{2c_j-1} e^{-\left(\frac{\lambda_j x_i^2}{\beta_j}\right)^{c_j}} \left[1 + c_j \left(ln\left(\frac{\lambda_j}{\beta_j}\right) + 2ln(x_i) - \left(\frac{\lambda_j x_i^2}{\beta_j}\right)^{c_j} ln\left(\frac{\lambda_j x_i^2}{\beta_j}\right)\right)\right]}{\sum_{j=1}^{k} 2\, p_j c_j \left(\frac{\lambda_j}{\beta_j}\right)^{c_j} x_i^{2c_j-1} e^{-\left(\frac{\lambda_j x_i^2}{\beta_j}\right)^{c_j}}} - \frac{(n-m)\, p_j\, e^{-\left(\frac{\lambda_j x_m^2}{\beta_j}\right)^{c_j}} \left(\frac{\lambda_j x_m^2}{\beta_j}\right)^{c_j} ln\left(\frac{\lambda_j x_m^2}{\beta_j}\right)}{\sum_{j=1}^{k} p_j\, e^{-\left(\frac{\lambda_j x_m^2}{\beta_j}\right)^{c_j}}}, \tag{36}$$

$$\frac{\partial \ell}{\partial \lambda_j} = \sum_{i=1}^{m} \frac{2p_j \frac{c_j^2}{\lambda_j} \left(\frac{\lambda_j}{\beta_j}\right)^{c_j} x_i^{2c_j-1} e^{-\left(\frac{\lambda_j x_i^2}{\beta_j}\right)^{c_j}} \left[1 - \left(\frac{\lambda_j x_i^2}{\beta_j}\right)^{c_j}\right]}{\sum_{j=1}^{k} 2\, p_j c_j \left(\frac{\lambda_j}{\beta_j}\right)^{c_j} x_i^{2c_j-1} e^{-\left(\frac{\lambda_j x_i^2}{\beta_j}\right)^{c_j}}} - \frac{(n-m)\, p_j \frac{c_j}{\lambda_j} \left(\frac{\lambda_j x_m^2}{\beta_j}\right)^{c_j} e^{-\left(\frac{\lambda_j x_m^2}{\beta_j}\right)^{c_j}}}{\sum_{j=1}^{k} p_j\, e^{-\left(\frac{\lambda_j x_m^2}{\beta_j}\right)^{c_j}}} \tag{37}$$

$$\frac{\partial \ell}{\partial \beta_j} = -\sum_{i=1}^{m} \frac{2p_j \frac{c_j^2}{\beta_j} \left(\frac{\lambda_j}{\beta_j}\right)^{c_j} x_i^{2c_j-1} e^{-\left(\frac{\lambda_j x_i^2}{\beta_j}\right)^{c_j}} \left[1 - \left(\frac{\lambda_j x_i^2}{\beta_j}\right)^{c_j}\right]}{\sum_{j=1}^{k} 2\, p_j c_j \left(\frac{\lambda_j}{\beta_j}\right)^{c_j} x_i^{2c_j-1} e^{-\left(\frac{\lambda_j x_i^2}{\beta_j}\right)^{c_j}}} + \frac{(n-m)\, p_j \frac{c_j}{\beta_j} \left(\frac{\lambda_j x_m^2}{\beta_j}\right)^{c_j} e^{-\left(\frac{\lambda_j x_m^2}{\beta_j}\right)^{c_j}}}{\sum_{j=1}^{k} p_j\, e^{-\left(\frac{\lambda_j x_m^2}{\beta_j}\right)^{c_j}}}. \tag{38}$$

Where $j = 1,2$, and $p_1 = p, p_2 = 1 - p$. The MLEs for each parameter can be derived either by solving the system of non-linear equations in (36), (37) and (38) numerically or by maximizing (35) by optimization techniques using the programming language R.

4.4. Estimation of the Reliability, Hazard and Reversed Hazard Rate Functions

The invariance property of the ML estimators enables us to obtain the ML estimators of reliability, hazard, and reversed hazard rate functions by replacing the parameters c_j, λ_j and β_j by their ML estimators in (28), (29) and (30) or (32), (33) and (34) or (36), (37) and (38), respectively, as follows:

$$\hat{R}(x) = pe^{-\left(\frac{\hat{\lambda}_1 x^2}{\hat{\beta}_1}\right)^{\hat{c}_1}} + (1-p)e^{-\left(\frac{\hat{\lambda}_2 x^2}{\hat{\beta}_2}\right)^{\hat{c}_2}}, \tag{39}$$

$$\hat{h}(x) = p\frac{2\hat{c}_1\,\hat{\lambda}_1^{\hat{c}_1}}{\hat{\beta}_1^{\hat{c}_1}}\,x^{2\hat{c}_1-1} + (1-p)\frac{2\,\hat{c}_2\,\hat{\lambda}_2^{\hat{c}_2}}{\hat{\beta}_2^{\hat{c}_2}}\,x^{2\hat{c}_2-1}, \tag{40}$$

$$\begin{aligned}\widehat{rh}(x) = {} & p\frac{\frac{2\hat{c}_1\,\hat{\lambda}_1^{\hat{c}_1}}{\hat{\beta}_1^{\hat{c}_1}}\,x^{2\hat{c}_1-1}e^{-\left(\frac{\hat{\lambda}_1 x^2}{\hat{\beta}_1}\right)^{\hat{c}_1}}}{1-e^{-\left(\frac{\hat{\lambda}_1 x^2}{\hat{\beta}_1}\right)^{\hat{c}_1}}} \\ & +(1-p)\frac{\frac{2\,\hat{c}_2\,\hat{\lambda}_2^{\hat{c}_2}}{\hat{\beta}_2^{\hat{c}_2}}\,x^{2\hat{c}_2-1}e^{-\left(\frac{\hat{\lambda}_2 x^2}{\hat{\beta}_2}\right)^{\hat{c}_2}}}{1-e^{-\left(\frac{\hat{\lambda}_2 x^2}{\hat{\beta}_2}\right)^{\hat{c}_2}}},\end{aligned} \tag{41}$$

where $x > 0$, $\hat{c}_j$, $\hat{\lambda}_j$, $\hat{\beta}_j > 0$,and $\hat{\lambda}_j$, $\hat{c}_j$and $\hat{\beta}_j$ are the ML estimators of λ_j, c_j, β_j and $j = 1, 2$.

5. Simulation study

In this section, a simulation study is proceeded to evaluate the performance of parameters' estimators of mixture Weibull Rayleigh distribution. Assuming the values of MWE parameters $c_1 = 15$, $\lambda_1 = 4, c_2 = 13$ and $\lambda_2 = 3$ with different sizes of sample ($n = 30, 50, 100$ and 150) and mixture weights ($p = 0.4$ and 0.5).

Table 1 and 2 report the averages of ML estimates, bias and MSE using maximum likelihood estimation method. β_1 and β_2 are supposed equal 1. It is noticed from these tables that the ML averages with full sample are very close to the initial values of the parameters as the sample size

increases. Also, Bias's and MSEs are decreasing when the sample size is increasing. This is indicative of the fact that the estimates are consistent and approaches the population parameter values as the sample size increases. Also it is noticed that the estimates are accurate at $p = 0.5$ more than at $p = 0.4$ according to the biases and MSEs.

Table 1. Monte Carlo Simulation Results for (MWR) Distribution: MLE, Bias and MSE for Full Sample at $p = 0.4$.

p	n	Results	Parameters			
			$c_1 = 15$	$\lambda_1 = 4$	$c_2 = 13$	$\lambda_2 = 3$
0.4	30	MLE	15.1470	3.9922	13.1328	2.9989
		Bias	0.1470	-0.0077	0.1328	-0.0010
		MSE	0.5502	0.0034	0.6738	0.0026
	50	MLE	15.0433	3.9960	13.0174	2.9979
		Bias	0.0433	-0.0039	0.0174	-0.0020
		MSE	0.2890	0.0008	0.4130	0.0004
	100	MLE	15.0253	3.9976	12.9541	2.999
		Bias	0.0253	-0.0023	-0.0458	-0.0009
		MSE	0.2123	0.0014	0.3181	0.0012
	150	MLE	15.0329	3.9992	13.0479	2.9990
		Bias	0.0329	-0.0007	0.0479	-0.0009
		MSE	0.1192	0.0002	0.1487	0.0001

Table 2. Monte Carlo Simulation Results for (MWR) Distribution: MLE, Bias and MSE for Full Sample at $p = 0.5$.

p	n	Results	Parameters			
			$c_1 = 15$	$\lambda_1 = 4$	$c_2 = 13$	$\lambda_2 = 3$
0.5	30	MLE	15.0967	3.9941	13.1279	2.9972
		Bias	0.0967	-0.0058	0.1279	-0.0027
		MSE	0.5597	0.0022	0.5901	0.0019
	50	MLE	15.0944	3.9960	13.0612	2.9984
		Bias	0.0944	-0.0039	0.0612	-0.0015
		MSE	0.5321	0.0007	0.5104	0.0005
	100	MLE	15.0444	3.9984	13.0141	2.9985
		Bias	0.0444	-0.0015	0.0141	-0.0014
		MSE	0.1779	0.0003	0.1659	0.0002
	150	MLE	15.0245	3.9995	13.0249	2.9988
		Bias	0.0245	-0.0005	0.0249	-0.0011
		MSE	0.1077	0.0002	0.1217	0.0001

Table 3 and 4 report the averages of ML estimates, bias and MSE with Type-I censoring data. In these tables, according to the biases and MSEs, it is noticed that the ML averages with Type-I censored samples are very close to the initial values of the parameters as the sample size increases. Also, Bias's and MSEs are decreasing when the sample size is increasing. This is indicative of the fact that the estimates are consistent and approaches the population parameter values as the sample size increases. In addition, it is observed that the estimates are accurate when the percentage is 90% compared with the other percentages 80% and 70% which is clear as sample size increases. This indicates that as the percent of censored data Type-I increases, the estimates become more accurate. Also, it is observed that the estimates are accurate when mixture weight $p = 0.5$ compared with $p = 0.4$.

Table 3. Monte Carlo Simulation Results for (MWR) Distribution: MLE, Bias and MSE for Type-I Censored at $p = 0.4$

p		n	Results	Parameters			
				$c_1 = 15$	$\lambda_1 = 4$	$c_2 = 13$	$\lambda_2 = 3$
0.4	90%	30	MLE	14.9990	4.0009	13.0002	3.0001
			Bias	-0.0009	0.0009	0.0002	0.0001
			MSE	0.0022	0.0016	0.0011	0.0012
		50	MLE	15.0001	4.0014	12.9991	3.0001
			Bias	0.0001	0.0014	-0.0008	0.0001
			MSE	0.00071	0.0004	0.0007	0.0004
		100	MLE	15.0000	4.0000	12.9997	3.0000
			Bias	0.0005	0.0002	-0.0003	0.0008
			MSE	0.0004	0.0003	0.0002	0.0002
		150	MLE	14.9991	4.0001	13.0009	3.0002
			Bias	-0.0001	0.0010	0.0010	0.0004
			MSE	0.0003	0.0002	0.0003	0.0003
	80%	30	MLE	15.0004	4.0002	12.9976	3.0000
			Bias	4.374e-04	2.409e-04	-2.359e-03	7.321e-05
			MSE	0.0014	0.0014	0.0012	0.0007
		50	MLE	14.9994	4.0017	13.0007	3.0022
			Bias	-0.0006	0.0017	0.0006	0.0022
			MSE	0.0009	0.0007	0.0007	0.0021
		100	MLE	14.9999	4.0013	12.9994	3.0007
			Bias	-8.521e-05	1.378e-03	-5.183e-04	7.209e-04
			MSE	0.0007	0.0004	0.0004	0.0003
		150	MLE	15.0008	4.0006	13.0004	3.0012
			Bias	0.0008	0.0006	0.0004	0.0012
			MSE	0.0004	0.0009	0.0008	0.0007
	70%	30	MLE	15.0027	4.0015	13.0016	3.0014
			Bias	0.0027	0.0015	0.0016	0.0014
			MSE	0.0051	0.0035	0.0037	0.0027
		50	MLE	15.0010	4.0019	12.9989	3.0002
			Bias	0.0011	0.0019	-0.0010	0.0002
			MSE	0.0016	0.0033	0.0015	0.0010
		100	MLE	14.9985	4.0015	12.9976	3.0045

			Bias	-0.0014	0.0015	-0.0023	0.0045
			MSE	0.0038	0.0029	0.0073	0.0119
		150	**MLE**	14.9992	4.0022	12.9994	3.0008
			Bias	-0.0008	0.0022	-0.0006	0.0009
			MSE	0.0010	0.0007	0.0007	0.0006

Table 4. Monte Carlo Simulation Results for (MWR) Distribution: MLE, Bias and MSE for Type-I Censored at $p = 0.5$.

p		n	Results	Parameters			
				$c_1 = 15$	$\lambda_1 = 4$	$c_2 = 13$	$\lambda_2 = 3$
0.5	**90%**	**30**	**MLE**	15.0002	4.0019	12.9987	2.9994
			Bias	0.0004	0.0019	-0.0012	0.0005
			MSE	0.0008	0.0009	0.0007	0.0006
		50	**MLE**	15.0001	4.0027	12.9992	3.0005
			Bias	0.0002	0.0027	-0.0008	0.0005
			MSE	0.0005	0.0005	0.0006	0.0005
		100	**MLE**	14.9994	4.0010	12.9986	2.9997
			Bias	-5.675e-04	1.063e-03	-1.393e-03	-2.860e-04
			MSE	0.0003	0.0005	0.0005	0.0004
		150	**MLE**	15.000	4.0009	12.9994	3.0003
			Bias	2.613e-05	9.727e-04	-5.902e-04	3.521e-04
			MSE	0.0004	0.0003	0.0003	0.0003
	80%	**30**	**MLE**	15.0003	4.0011	13.0002	3.0009
			Bias	0.0004	0.0002	0.0012	0.0010
			MSE	0.0012	0.0008	0.0019	0.0018
		50	**MLE**	14.9983	4.0069	12.9989	3.0053
			Bias	-0.0017	0.0069	-0.0011	0.0053
			MSE	0.0031	0.0019	0.00583	0.0024
		100	**MLE**	15.0003	4.0013	13.0004	2.9998
			Bias	0.0003	0.00137	0.00046	-0.00015
			MSE	0.0006	0.0004	0.0006	0.0004
		150	**MLE**	15.0008	4.0019	12.9997	3.0003
			Bias	0.0008	0.0020	-0.0003	0.0003
			MSE	0.0004	0.0004	0.0005	0.0003
	70%	**30**	**MLE**	14.9990	3.9998	12.9968	3.0020
			Bias	-9.968e-05	-1.048e-04	-3.128e-03	2.050e-03
			MSE	0.0016	0.0032	0.0018	0.0015
		50	**MLE**	14.9986	4.0022	12.9999	2.9991
			Bias	-1.378e-03	2.295e-03	-1.682e-05	-8.877e-04
			MSE	0.0009	0.0009	0.0016	0.0010
		100	**MLE**	14.9994	4.0031	12.9997	3.0007

			Bias	-0.0006	0.0031	-0.00028	0.0007
			MSE	0.0007	0.0005	0.0007	0.0006
		150	**MLE**	15.0014	4.0047	13.0002	2.9987
			Bias	0.0013	0.0047	0.0002	-0.0012
			MSE	0.00108	0.0013	0.0009	0.0005

Table 5. Monte Carlo Simulation Results for (MWR) Distribution: MLE, Bias and MSE for Type II Censored at $p = 0.4$.

p		n	Results	Parameters			
				$c_1 = 15$	$\lambda_1 = 4$	$c_2 = 13$	$\lambda_2 = 3$
0.4	**90%**	**30**	**MLE**	15.1479	3.9904	13.1298	2.9971
			Bias	0.1479	-0.0095	0.1298	-0.0028
			MSE	0.3306	0.0022	0.6365	0.0008
		50	**MLE**	15.0848	3.9946	13.0867	2.9981
			Bias	0.0848	-0.0053	0.0867	-0.0018
			MSE	0.2522	0.0013	0.4871	0.0004
		100	**MLE**	15.0714	3.9981	12.9754	2.9980
			Bias	0.0714	-0.0018	-0.0245	-0.0019
			MSE	0.3212	0.0005	0.4316	0.0002
		150	**MLE**	15.0336	3.9987	13.0402	2.9990
			Bias	0.0336	-0.0012	0.0402	-0.0009
			MSE	0.0781	0.0003	0.1208	0.0001
	80%	**30**	**MLE**	15.1000	3.9739	13.0729	2.9985
			Bias	0.1000	-0.0260	0.0729	-0.0014
			MSE	0.1519	0.0115	0.5132	0.0027
		50	**MLE**	15.0655	3.9891	13.0768	2.9980
			Bias	0.0655	-0.0108	0.0768	-0.0019
			MSE	0.0956	0.0034	0.3604	0.0004
		100	**MLE**	15.0316	3.9974	13.0115	2.9983
			Bias	0.0316	-0.0025	0.0115	-0.0016
			MSE	0.1040	0.0008	0.1392	0.0002
		150	**MLE**	15.0159	3.9986	13.0103	2.9989
			Bias	0.0159	-0.0013	0.0103	-0.0010
			MSE	0.0386	0.0005	0.1141	0.0001
	70%	**30**	**MLE**	15.0979	3.9250	13.1765	2.9972
			Bias	0.0979	-0.0749	0.1765	-0.0027
			MSE	0.1122	0.0394	0.9171	0.0027
		50	**MLE**	15.0741	3.9368	13.0819	2.9974
			Bias	0.0741	-0.0631	0.0819	-0.0025
			MSE	0.0688	0.0369	0.6842	0.0005

		100	**MLE**	15.0199	3.9801	13.0004	2.9979
			Bias	0.0199	-0.0198	0.0004	-0.0020
			MSE	0.0156	0.0099	0.1581	0.0002
		150	**MLE**	15.0203	3.9937	13.0183	2.9989
			Bias	0.0203	-0.0062	0.0183	-0.0010
			MSE	0.0135	0.0018	0.0792	0.0001

Table 6. Monte Carlo Simulation Results for (MWR) Distribution: MLE, Bias and MSE for Type II Censored at $p = 0.5$.

p		n	**Results**	**Parameters**			
				$c_1 = 15$	$\lambda_1 = 4$	$c_2 = 13$	$\lambda_2 = 3$
0.5	**90%**	**30**	**MLE**	15.1210	3.9916	13.1442	2.9973
			Bias	0.1210	-0.0083	0.1442	-0.0026
			MSE	0.5301	0.0026	0.6562	0.0020
		50	**MLE**	15.0956	3.9932	13.0932	3.0005
			Bias	0.0956	-0.0067	0.0932	0.0005
			MSE	0.4461	0.0029	0.5882	0.0025
		100	**MLE**	15.1010	3.9980	13.0839	2.9986
			Bias	0.1010	-0.0019	0.0839	-0.0013
			MSE	0.3772	0.0004	0.4883	0.0003
		150	**MLE**	15.0220	3.9991	13.0307	2.9988
			Bias	0.0220	-0.0008	0.0307	-0.0011
			MSE	0.0935	0.0002	0.1329	0.0001
	80%	**30**	**MLE**	15.1777	3.9878	13.1233	2.9979
			Bias	0.1777	-0.0121	0.1233	-0.0020
			MSE	0.7314	0.0046	0.8392	0.0027
		50	**MLE**	15.0936	3.9935	13.0788	2.9985
			Bias	0.0935	-0.0064	0.0788	-0.0014
			MSE	0.3279	0.0013	0.4579	0.0005
		100	**MLE**	15.1045	3.9961	13.1073	2.9998
			Bias	0.1045	-0.0038	0.1073	-0.0001
			MSE	0.4475	0.0016	0.3934	0.0013
		150	**MLE**	15.0051	3.9991	13.0183	2.9988
			Bias	0.0051	-0.0008	0.0183	-0.0011
			MSE	0.0762	0.0003	0.0914	0.0001
	70%	**30**	**MLE**	15.1846	3.9697	13.1297	2.9983
			Bias	0.1846	-0.0302	0.1297	-0.0016
			MSE	0.4863	0.0161	0.8242	0.0035
		50	**MLE**	15.0871	3.9845	13.0877	2.9992
			Bias	0.0871	-0.0154	0.0877	-0.0007
			MSE	0.2909	0.0058	0.5270	0.0015

		100	**MLE**	15.0390	3.9947	13.0292	2.9994
			Bias	0.0390	-0.0052	0.0292	-0.0005
			MSE	0.1363	0.0020	0.2230	0.0012
		150	**MLE**	15.0336	3.9970	13.0170	2.9998
			Bias	0.0336	-0.0029	0.0170	-0.0001

6. Applications

The main aim of this subsection is to evaluate the performance of the proposed MWR through two real data sets. The MWR distribution is fitted to the two real data sets in comparison with other distributions with the following density functions using the R programming language:

1- K
umaraswamy Weibull (KW) distribution by Hassan and Elgarhy [13]

$$f(x)_{Ku-} \quad = \alpha\beta\frac{c}{\gamma}\left(\frac{x}{\gamma}\right)^{c-1} e^{-\left(\frac{x}{\gamma}\right)^{c}}\left[1-e^{-\left(\frac{x}{\beta}\right)^{c}}\right]^{\alpha-1}\left[1-\left(1-e^{-\left(\frac{x}{\beta}\right)^{c}}\right)^{\alpha}\right]^{\beta-1}, \tag{42}$$

where $x, \alpha, \beta, c, \gamma > 0$.

2- B
eta Weibull (BW) distribution by Lee et al. [14].

$$f(x)_{BW} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{c}{\gamma}\left(\frac{x}{\gamma}\right)^{c-1}\left[1-e^{-\left(\frac{x}{\gamma}\right)^{c}}\right]^{\alpha-1} e^{-\beta\left(\frac{x}{\gamma}\right)^{c}}, \tag{43}$$

where $x, \alpha, \beta, c, \gamma > 0$.

3- W
eibull (W) distribution by Lai et al. [15].

$$f_W(x) = \frac{a}{\gamma}\left(\frac{x}{\gamma}\right)^{a-1} e^{-\left(\frac{x}{\gamma}\right)^{a}}, \tag{44}$$

where $x \geq 0, a, \gamma > 0$.

4- R
ayleigh Distribution.

$$p(E) = \frac{2E}{s} e^{-\frac{E^2}{s}}, \tag{46}$$

where $s = \langle E^2 \rangle = nE_j^2$ and the brachets $\langle\ \rangle$ denote the mean value.

5- W
eibull Exponential (WE) distribution.

$$f(x) = \frac{1}{\beta^c} c r^c x^{c-1} e^{-\left(\frac{rx}{\beta}\right)^{c}},$$

where $x \geq 0, c, r, \beta > 0$.

6- T
he alpha power exponentiated Weibull-exponential distribution (APEWED) by Akattawi and Aljuhani [16].

$$f(x) = \begin{cases} \frac{\log\alpha}{\alpha-1}\frac{ac}{\gamma}\frac{\lambda e^{-\lambda}\left(1-e^{-\lambda x}\right)^{c-1}}{1-(1-e^{\lambda x})^{c}}\left(\frac{-\log\left(1-\left(1-e^{-\lambda x}\right)^{c}\right)}{\gamma}\right)^{a-1} e^{-\left(\frac{-\log\left(1-\left(1-e^{-\lambda x}\right)^{c}\right)}{\gamma}\right)} \alpha^{1-e^{-\left(\frac{-\log\left(1-\left(1-e^{-\lambda x}\right)^{c}\right)}{\gamma}\right)^{a}}} & if\ \alpha > 0, \alpha \neq 1 \\ \frac{ac}{\gamma}\frac{\lambda e^{-\lambda}\left(1-e^{-\lambda x}\right)^{c-1}}{1-(1-e^{\lambda x})^{c}}\left(\frac{-\log\left(1-\left(1-e^{-\lambda}\right)^{c}\right)}{\gamma}\right)^{a-1} e^{-\left(\frac{-\log\left(1-\left(1-e^{-\lambda x}\right)^{c}\right)}{\gamma}\right)^{a}} & if\ \alpha = 1. \end{cases} \tag{45}$$

The first data set is given by Bourguignon et al. [12]. The data refers to the strengths of 1.5 cm glass fibres, originally obtained by workers at the UK National Physical Laboratory. The data set consists of 63 observations as follows:

0.55, 0.74, 0.77, 0.81, 0.84, 0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49, 1.49, 1.50, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61, 1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68, 1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76, 1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24.

The second data was obtained from Hogg and Klugman [18]. This data represents the losses due to windrelated catastrophes. The sorted values include claims of 2,000,000 and for convenience they have been recorded in millions, described as:

2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 6, 8, 8, 9, 15, 17, 22, 23, 24, 24, 25, 27, 32, 43.

The MLEs of parameters and the corresponding standard errors (SEs), for all fitted distributions to the real data sets are reported in tables 7 and 8 respectively.

In order to compare between the distributions, different criteria are considered such as (log-likelihood), Akaike information criterion (AIC), Akaike information criterion corrected (AICC), Bayesian information criterion (BIC) and Hannan -Quinn information criterion (HQIC) where:

$AIC = -2\ log\ L(\theta) + 2\ v,$

$$AICC = AIC + \frac{2v^2 + 2v}{n - v - 1}\ ,$$

$$BIC = -2\ log\ L(\theta) + v\ log(n),$$

$$HQIC = -2\ log\ L(\theta) + 2v\ \ log(log\ \ n)\ ,$$

where v denotes the number of parameters in the model and n denotes the number of observations. The best distribution corresponds the lowest values of these measures.

In addition, different goodness-of-fit measures such as the Kolmogorov–Smirnov (KS) test, the Anderson Darling (AD) test and Cramer Von-Messes (CM) test are applied to check the validity of the fitted model. The KS, AD and CM test statistics and the relevant p-values for different models with respect to the three real data sets are also reported in tables 9 and 10.

Table 7. The MLEs (SEs in parentheses) for first data set.

Distribution	Estimated Parameters					
WRM $(\hat{c}_1, \hat{\beta}_1, \hat{\lambda}_1, \hat{c}_2, \hat{\beta}_2, \hat{\lambda}_2)$	1.8811 (0.2979)	3.2550 (8.2114)	0.6045 (0.7627)	7.5250 (1.5007)	3.2894 (2.5805)	0.6506 (0.2553)
KW $(\hat{\alpha}, \hat{\beta}, \hat{c}, \hat{\gamma})$	0.5602 (0.1610)	0.2058 (0.0320)	6.7861 (0.3020)	1.3246 (0.0035)		
BW $(\hat{\alpha}, \hat{\beta}, \hat{c}, \hat{\gamma})$	0.5832 (0.1192)	0.2132 (0.0342)	7.3058 (0.3238)	1.3719 (0.0036)		
APEWED $(\hat{\alpha}, \hat{a}, \hat{c}, \hat{\gamma}, \hat{\lambda})$	3.3254 2.8898	4.1670 1.2421	1.5415 (3.5943)	1.5992 (1.2354)	1.2932 (2.8110)	
WEG $(\hat{c}, \hat{r}, \hat{\beta})$	5.7807 (0.5761)	6.8460 (898.1364)	4.204864 (551.6424)			
W $(\hat{\alpha}, \hat{\gamma})$	1.6281 (0.0371)	5.780705 (0.5761)				
R $(\hat{\gamma})$	1.0895 (0.0686)					

Table 8. The MLEs (SEs in parentheses) for second data set.

Distribution	Estimated Parameters					
WRM $(\hat{c}_1, \hat{\beta}_1, \hat{\lambda}_1, \hat{c}_2, \hat{\beta}_2, \hat{\lambda}_2)$	0.8729 (0.3232)	8.7899 (4.5881)	5.0398 (0.3481)	1.0137 (0.1917)	0.1729 (0.1486)	6.1400 (2.6909)
KW $(\hat{\alpha}, \hat{\beta}, \hat{c}, \hat{\gamma})$	0.5829 (0.0991)	0.1154 (0.0189)	0.7944 (0.0026)	0.4741 (0.0026)		
BW $(\hat{\alpha}, \hat{\beta}, \hat{c}, \hat{\gamma})$	1.7457 (0.6348)	0.1479 (0.0257)	1.0660 (0.0034)	1.4967 (0.0034)		
APEWED $(\hat{\alpha}, \hat{a}, \hat{c}, \hat{\gamma}, \hat{\lambda})$	0.3484 (1.7147)	0.7925 (0.6759)	1.8655 (3.0664)	1.0594 (2.6533)	0.1375 (0.5290)	
WEG $(\hat{c}, \hat{r}, \hat{\beta})$	1.0095 (0.1219)	8.4865 (1.4950)	0.8454 (0.0026)			
W $(\hat{\alpha}, \hat{\gamma})$	9.5067 (1.6367)	1.0016 (0.1211)				
R $(\hat{\gamma})$	9.9055 (0.8034)					

Table 9. The goodness-of-fit measures for first data set.

Distribution	Statistic

	Log.Lik	AICC	A I C	B I C	HQIC	K S P	KS.ST	CV.P	CV.ST	AN.S	AN.ST
WRM	-9.2124	31.9247	30.4247	43.2835	35.4821	0.8416	0.0777	0.9939	0.1856	0.9732	0.0311
KW	13.6821	36.0539	35.3643	43.9368	38.7359	0.1843	0.1375	0.4042	0.9158	0.3328	0.1707
BW	14.0350	36.7597	36.0701	44.6426	39.4417	0.2372	0.1300	0.3987	0.9250	0.3487	0.1646
APEWED	14.2960	39.6446	38.5920	49.3077	42.8065	0.1250	0.1483	0.3025	1.1137	0.2734	0.1972
WEG	15.207	36.820	36.414	42.843	38.942	0.10784	0.1522	0.0000	20.957	$9.5 x 10^{-6}$	$4.3 x 10^{2}$
W1	15.2068	34.6137	34.4137	38.7000	36.0995	0.1078	0.1522	0.2524	1.2407	0.2403	0.2151
R1	49.791	101.654	101.58	103.72	102.42	$1.5 x 10^{-6}$	0.3339	$9.7 x 10^{-6}$	11.425	$1.6 x 10^{-6}$	2.3221

Table 10. The goodness-of-fit measures for second data set.

Distribution	Statistic										
	Log.Lik	AICC	A I C	B I C	HQIC	K S P	KS.ST	CV.P	CV.ST	AN.S	AN.ST
WRM	115.658	246.026	243.316	253.141	246.811	0.1826	0.1774	0.1871	1.4571	0.2231	0.2257
KW	127.54	264.29	263.07	269.62	265.40	$2.3 x 10^{-4}$	0.3449	0.0135	3.6231	0.02031	0.6126
BW	122.587	254.385	253.173	259.723	255.503	0.0237	0.2416	0.0133	3.6348	0.0182	0.6318
APEWED	121.176	254.227	252.352	260.539	255.265	0.2116	0.1719	0.1276	1.7453	0.1777	0.2587
WEG	123.26	253.23	252.53	257.44	254.27	0.07106	0.20955	0.0000	12.667	$1.5 x 10^{-5}$	–
W1	123.549	251.441	251.098	254.373	252.263	0.1282	0.1901	0.0860	2.0554	0.1024	0.3430
R1	146.29	294.68	294.57	296.21	295.15	$2.3 x 10^{-8}$	0.4902	$1.5 x 10^{-5}$	20.727	$6.2 x 10^{-8}$	2.7611

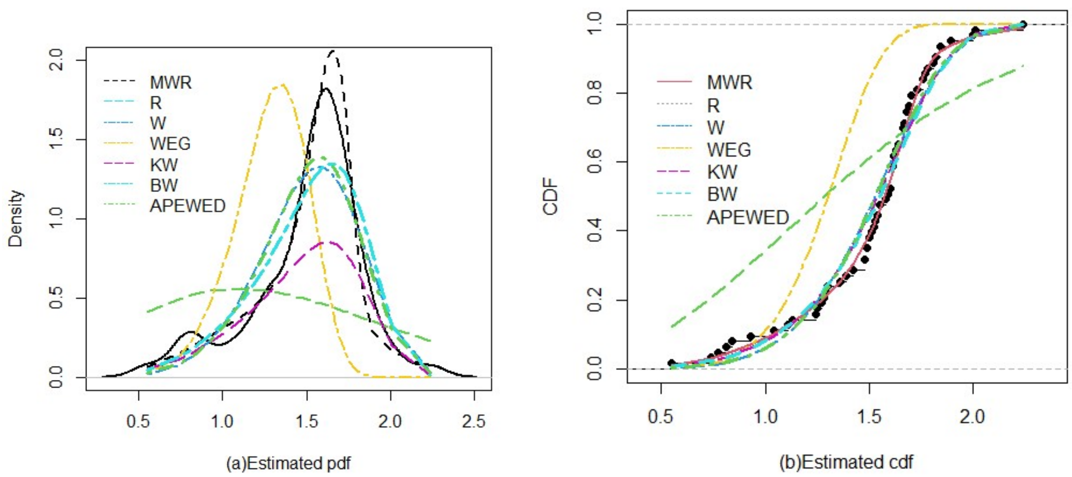

Fig. 2 Observed and expected frequencies for each model for data set 1.

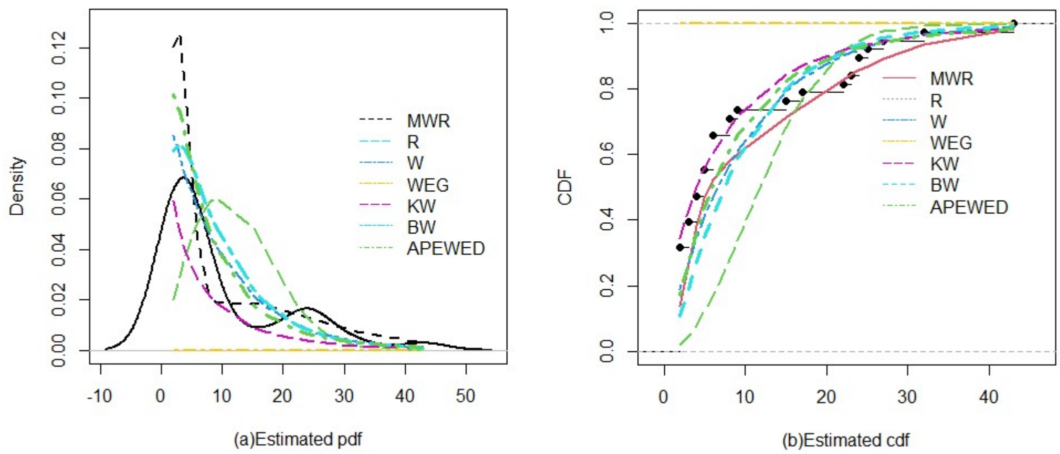

Fig.3 Observed and expected frequencies for each model for data set 2.

From tables 9 and 10, it can be seen that the MWR had the lowest values of log likelihood, AICC, AIC, BIC and HQIC values, as well as the best KS, AD and CM statistics values and p-values, which means that the MWR provided the most proper fit to the three sets of data, as compared to the other models

Figures 2 and 3 show the PDF and CDF of the models for the two data sets. It can be seen, from Figures 2 and 3, that the MWR provided the closest fit to the observed distribution (i.e., sold line) for the two data sets.

7. Conclusions

In this study, the MWR distribution was introduced based on mixture approach in order to provide flexibility in fitting different types of data. General statistical properties were obtained. The maximum likelihood estimation method was employed for estimating the parameters of the proposed distribution based on complete samples, Type-I and Type-II censored samples. The performances of these MLEs were tested through simulation studies. Three real data sets were considered in order to assess the applicability of the proposed distribution comparing to other distributions. The results indicate that the introduced distribution MWR can offer the best fit compared to the competing distributions.

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