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UNIQUE FIXED POINT THEOREMS USING SELF MAPPING IN 2 METRIC SPACE

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Abstract

In this paper, we have obtained some fixed point theorems in 2-metric space using self mapping and also our results also generalizes an existing result in 2-metric spaces.

Keywords: Fixed point, 2-metric space, Completeness, Self mapping

Mathematical Subject Classification : 54H25, 47H10,

1. Introduction

The concept of 2-metric space was initiated by S. Gahler [1,2]. The study was further enhanced by B.E. Rhoades [6], Iselei [3], Miczko and Palezewaki [4] and Saha and Day [7], Khan [5], Saha and Baisnab [8]. Moreover B.E. Rhoades and other introduced several properties of 2-metric spaces and proved some fixed point and common fixed point theorems for contractive and expansion mappings and also have found some interesting results in 2-metric space, where in each cases the idea of convergance of sum of a finite or infinite series of real constants plays a crucial role in the proof of fixed point theorems. In this same way, we prove a fixed point theorem and common fixed point theorems for the mapping satisfying different types of contractive conditions in 2-metric space and our results based on Saha and Day [9] .

2. Definition and Preliminaries

Definition 2.1

Let X be a non empty set. A real valued function d on $X \times X \times X$ is said to be a 2-metric in X if

(i) To each pair of distinct points x, y in X. There exists a point $z \in X$ such that $d(x, y, z) \neq 0$

(ii) $d(x, y, z) = 0$, when at least of x, y, z are equal.

(iii) $d(x, y, z) = d(y, z, x) = d(x, z, y)$

(iv) $d(x, y z) \leq d(x, y, w) + (x, w, z) + d(w, y, z)$ for all $x, y, z, w \in X$

when d is a 2-metric an X, then the ordered pair (X,d) is called is 2-metric space.

Definition 2.2

A sequence in 2-metric space (X,d) is said to be convergent to an element $x \in X$ if $\lim_{n\to\infty} d(x_n, x, a) = 0$ for all $x \in X$.

It follows that if the sequence $\{x_n\}$ converges to x then $\lim_{n\to\infty} d(x_n, a, b,) = d(x, a, b)$ for all a b $\in$ X.

Definition 2.3

A sequence in a 2-metric space X is Cauchy sequence if $\lim_{x,n\to\infty} d(x_m,x_n,a)=0$ for all $x \in X$.

Definition 2.4

If a sequence is convergent in a 2-metric space then it is a Cauchy Sequence.

Definition 2.5

A 2-metric space (X,d) is said to be complete if every Cauchy sequence in X is convergent in the same space.

Lemma 2.6

If a sequence $\{x_n\}$ in a 2-metric space converges to *x* there every subsequence of $\{x_n\}$ also converges to the same limit *x*.

Lemma 2.7

Limit of a sequence in a 2-metric space, if exists, is unique.

3. Main Results

Theorem 3.1

Let (X,d) be a complete 2-metric space. Let T be a self map on X satisfying the conditions:

$$d(T^i x, T^i y, a) \le \alpha_i [d(x, Tx, a) + d(y, Ty, a)] + \beta_i [d(x, Tx, a) + d(y, Tx, a)] + \gamma_i d(x, y, a) \quad (1)$$

for all x, y, a $\in$ X and

Let $0 \le \alpha_i$, $0 \le \beta_i < 1$, $0 \le \gamma_i < 1$ (i = 1, 2,....)

with $\sum_{i=1}^{\infty}(\alpha_i + \beta_i + \gamma_i) < \infty$, Then T has a unique fixed point in X.

Proof:

For any $x \in X$, Let $x_n = T^n(x)$ then

$d(Tx_0, T^2x_0, a) = d(Tx_0, TTx_0, a)$

Using (1), we have

$\leq \alpha_1 [d(x_0, Tx_0, a) + d(Tx_0\ T^2x_0\ a)]$

$+ \beta_1 [d(x_0, Tx_0, a) + d(Tx_0\ Tx_0\ a)] + \gamma_1 d(x_0, Tx_0, a)]$

$= \alpha_1 [d(x_0, Tx_0, a) + \alpha_1 d(Tx_0\ T^2x_0\ a)$

$+ \beta_1 d(x_0, Tx_0, a) + \beta_1 d(x_1, x_1\ a) + \gamma_1 d(x_0, Tx_0, a)$

$(1-\alpha_1)d(Tx_0, T^2x_0, a) \leq (\alpha_1 + \beta_1 + \gamma_1)\, d(x_0, Tx_0, a)$

$$d(Tx_0, T^2x_0, a) \leq \left(\frac{\alpha_1+\beta_1+\gamma_1}{1-\alpha_1}\right) d(x_0, Tx_0, a) \qquad (2)$$

Now

$d(x_n, x_{n+1}, a) = d(T^nx_0, T^{n+1}x_0, a)$

$= d(T^nx_0, T^nT(x_0, a)$

$\leq \alpha_n [d(x_0, Tx_0, a) + d(Tx_0\ T^2x_0\ a)]$

$+ \beta_n [d(x_0, Tx_0, a) + d(Tx_0\ Tx_0\ a)] + \gamma_n d(x_0, Tx_0, a)]$

$= \alpha_n [d(x_0, Tx_0, a) + \alpha_n d(Tx_0\ T^2x_0\ a) + \beta_n d(x_0, Tx_0, a) + \gamma_n d(x_0, Tx_0\ a)$

$$\alpha_n d(x_0, Tx_0, a) + \alpha_n\left(\frac{\alpha_1+\beta_1+\gamma_1}{1-\alpha_1}\right) d(x_0, Tx_0, a) + \beta_n d(x_0, Tx_0\ a) + \gamma_n d(x_0, Tx_0, a)$$

$$\leq \alpha_n\left(\frac{\alpha_1+\beta_1+\gamma_1}{1-\alpha_1}\right) d(x_0, Tx_0, a) + (\beta_n + \gamma_n)\, d(x_0, Tx_0, a) \qquad \text{by (2)}$$

$$d(x_n, x_{n+1}, a) \leq \left\{\left(\frac{1+\beta_1+\gamma_1}{1-\alpha_1}\right)\alpha_n + (\beta_n+\gamma_n)\right\} d(x_0, Tx_0, a) \qquad (3)$$

any positive integer p,

$$d(x_n, x_{n+p}, a) = d(x_{n+p}, x_n, a) \leq \sum_{k=0}^{p-2} d(x_{n+p}, x_{n+k}, x_{n+k+1}) + \sum_{k=0}^{p-1} d(x_{n+k}, x_{n+k+1}, a) \quad (4)$$

Now

$$\sum_{k=0}^{p-2} d(x_{n+p}, x_{n+k}, x_{n+k+1}) = d(x_{n+p}, x_n, x_{n+1}) + d(x_{n+p}, x_{n+1}, x_{n+2}) + \ldots$$

$$\leq \left\{\left(\frac{1+\beta_1+\gamma_1}{1-\alpha_1}\right)\alpha_n + (\beta_n+\gamma_n)\right\} d(x_0, Tx_0, x_{n+p})$$

$$+\left\{\left(\frac{1+\beta_1+\gamma_1}{1-\alpha_1}\right)\alpha_{n+1}+\left(\beta_n+\gamma_{n+1}\right)\right\} d\,(x_0, Tx_0, x_{n+p})$$

$+ \ldots..$ by (3)

$d\,(x_0, T(x_0), x_{n+p}) = d\,(Tx_{n+p+1}, Tx_0, x_0)$

$\leq \alpha_1\,[d(x_{n+p-1}\ Tx_{n+p-1}, x_0) + d\,(x_0, Tx_0, x_0)]$

$+ \beta_1\,[dx_{n+p-1}, Tx_{n+p-1}, x_0] + d\,(x_0, Tx_{n+p-1}, x_0) + \gamma_1\, d(x_{n+p-1}, x_0, x_0)$

$= \alpha_1\, d(x_{n+p-1}, x_{n+p}, x_0) + \beta_1\, d(x_{n+p-1}, x_{n+p}, x_0)$

$= (\alpha_1 + \beta_1)\, d(x_{n+p-1}, x_{n+p}, x_0)$

put $n + p - 1 = m$ then

$d(x_0, Tx_0, x_{n+p}) \leq (\alpha_1 + \beta_1)\, d(x_m, x_{m+p}, x_0)$

$$\leq (\alpha_1+\beta_1)\left\{\left(\frac{1+\beta_1+\gamma_1}{1-\alpha_1}\right)\alpha_m+\beta_m+\gamma_m\right\} d\,(x_0, x_1, x_0)$$

$= 0$ by (3)

$$\sum_{k=0}^{p-2} d(x_{n+p}, x_{n+k}, x_{n+k+1}) = 0$$

then from (4), we have

$$d(x_n, x_{n+p}, a) = d(x_{n+p}, x_n, a) \leq \sum_{k=0}^{p-1} d(x_{n+k}, x_{n+k+1}, a)$$

$$\leq \sum_{k=0}^{p-1} d(x_{n+k}, x_{n+k+1}, a)$$

$$\leq \sum_{k=0}^{p-1}\left\{\left(\frac{1+\beta_1+\gamma_1}{1-\alpha_1}\right)\alpha_{n+k}+\beta_{n+k}+\gamma_{n+k}\right\} d\,(x_0, Tx_0, a) \quad \text{by (3)}$$

$$d(x_{n+p}, x_n, a) \leq \sum_{k=0}^{p-1} d(x_{n+k}, x_{n+k+1}, a)$$

$$\leq \sum_{k=0}^{p-1}\left\{\left(\frac{1+\beta_1+\gamma_1}{1-\alpha_1}\right)\alpha_{n+k}+\beta_{n+k}+\gamma_{n+k}\right\} d\,(x_0, Tx_0, a)$$

$$\leq \left\{\left(\frac{1+\beta_1+\gamma_1}{1-\alpha_1}\right)\sum_{k=0}^{p-1}\alpha_{n+k}+\sum_{k=0}^{p-1}\beta_{n+k}+\sum_{k=0}^{p-1}\gamma_{n+k}\right\} d\,(x_0, Tx_0, a)$$

Now Since $\Sigma\,(\alpha_n + \beta_n + \gamma_n) < \infty$

$d(x_{n+p}, x_n, a) \to 0$ as $n \to \infty$

So $\{x_n\}$ is a Cauchy sequence in X and by completeness of X

Hence $\{x_n\}$ converges to a point $u \in X$.

Again

$d(x_{n+1}, Tu, a) = d(T^{n+1} x_0, Tu, a)$

$= d(TT^n x_0, Tu, a)$

$\leq \alpha_1[d(T^n x_0, T^{n+1} x_0, a) + d(u, Tu, a]$

$+ \beta_1[d(T^n x_0, T^{n+1}(x_0), a) + d(u, T^{n+1}(x_0), a]$

$+ \gamma_1(T^n x_0, u, a)$

$\leq \alpha_1 d(x_0\ x_{n+1}\ a) + \alpha_1 d(u, Tu, a)$

$+ \beta_1 d(x_n, x_{n+1}, a) + \beta_1 d(u, x_{n+1}, a) + \gamma_1(x_n, u, a)$

Taking limit as $n \to \infty$, we get

$d(u, Tu, a) \leq \alpha_1 d(u, Tu\ a)$

$d(u, Tu, a) \leq \alpha_1 d(u, Tu\ a)$

Thus $Tu = u$

Uniqueness, let u, v be two fixed points of T.

Then

$d(u, v, a) = d(Tu\ Tv, a) \leq \alpha_1 d(u, Tu\ a)$

$+ \alpha_1 d(v, Tv\ a) + \beta_1 d(u, Tu\ a)$

$+ \beta_1 d(v, Tu\ a) + \gamma_1 d(u, v, a)$

$d(u, v, a) \leq (\beta_1 + \gamma_1) d(u, v, a)$

$d(u, v, a) \leq (\beta_1 + \gamma_1) d(u, v, a)$ as $0 \leq \beta_1 < 1, 0 \leq \gamma_1 < 1$

$\Rightarrow u = v$

Theorem 3.2

Let (X,d) be a complete 2-metric space, let T be a self mapping on X satisfying the conditions

$d(T^i x, T^i y, a) \leq \alpha_i [d(x, Tx, a) + d(y, Ty, a)]$

$+ \beta_i [d(x, Tx, a) + d(y, Tx, a)] + \gamma_i d(x, y, a)$

for all $x, y, a \in X$; and $0 \leq \alpha_i$, $0 \leq \beta_i < 1$, $0 \leq \gamma_i < 1$ (i = 1, 2, ...) with $\sum_{i=1}^{\infty}(\alpha_n + \beta_n + \gamma_n) < \infty$ If for some $x \in X$, $\{T^n(x)\}$ has a subsequence $\{T^{n_k}(x)\}$ with $\lim_{k\to\infty}\{T^{n_k}(x)\} = u \in X$ Then u is the unique fixed point of T.

Proof. we have for $x, a \in X$.

$d(u, Tu, a) \leq d(u, Tu, T^{n_k+1}x + d(u, T^{n_k+1}x, a)$

$$+ d(T^{n_k+1}x, Tu, a) \quad (5)$$

$d(T^{n_k+1}x, Tu, a) = d(T\,T^{n_k} x, Tu, a)$

$\leq \alpha_1 [d(T^{n_k} x, T^{n_k+1}x, a) + d(u, Tu, a)] + \beta_1[d(T^{n_k} x, T^{n_k+1}x, a)$

$+ d(u, T^{n_k+1}x, a] + \gamma_1 d(T^{n_k} x, u, a)$

Now from (5)

$d(u, Tu, a) \leq d(u, Tu, T^{n_k+1}x) + d(u, T^{n_k+1}x, a)$

$+\alpha_1[d(T^{n_k} x, T^{n_k+1}x, a) + d(u), T(u), a)] + \beta_1[d(T^{n_k}(x), T^{n_k+1}(x), a)$

$+ d(u, T^{n_k+1} x, a] + \gamma_1 d(T^{n_k} x, u, a)$

Taking limit $k \to \infty$, we get

$d(u, Tu, a) \leq \alpha_1 d(u, Tu, a)$

Thus $d(u, Tu, a) = 0$

Hence $u = Tu$.

Remark:

1. If $\alpha_i = \beta_i$ and $\beta_i = 0$ in (1) we get corresponding results of Saha and Day [9].
2. Hence our results generalize form of the results Saha and Day [9].

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