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POWER GRAPHS OF NON-GROUP SEMIGROUPS OF ORDER $p^{\alpha} q^{\beta}$

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Abstract
For every different odd primes p and q we attempt to construct a class of non- group semigroups of order $p^{\alpha}q^{\beta}$ by using the semidirect product of monogenic semigroups of indices greater than 1. Necessary conditions are given for the power graphs of monogenic and constructed semigroups to be Eulerian or complete.
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1 Introduction

Because of many interesting applications of the fnite semigroups in mathematics, computer science and fnite machines, constructing any subclass of non-group semigroups is of interest especially when they are non-commutative. In this paper we intend to construct a class of such semigroups by using the semidirect product of monogenic semigroups of indices greater than 1. The undirected power graphs of considered semigroups will be studied as well. Following [1, 2, 5, 6, 9] we recall the defnition of undirected power graph $P(S)$, for an algebraic structure S. The vertex set of $P(S)$ is S and two vertices x and y are adjacent if and only if $x = y^m$ or $y = x^m$, for some integer $m \geq 2$. Following [10, 12] and present the defnition of semidirect product of two semigroups. For two semigroups S,T and a homomorphism $\phi : T \to End(S)$ the semidirect product of S by T, denoted by $S \rtimes_{\phi} T$ is a semigroup consists of the ordered pairs (s,t) where, $s \in S$ and $t \in T$ such that the multiplication is defned by:
$(s,t)(s',t') = (s\phi_t(s'),tt'), \phi(t) = \phi_t \in End(S)$
For all $s,s' \in S$ and $t,t' \in T$.

As of the last notation on the semigroups we follow [3, 4, 7, 8, 11]. The preliminaries on the semigroup theory may be found in [7, 8]. For detailed information on the semigroup (or monoid) presentations one may consult [3, 4, 11]. We prefer to give a brief history on the fnitely presented semigroups and monoids. A semigroup (or monoid) S is said to be presented by a semigroup (or monoid) presentation $\langle A|R\rangle$ if $S =\sim A^{+}/\rho (or S =\sim A^{*}/\rho)$ where, A is an alphabet, A^{+} is the free semigroup on A, $A^{*} = A \cup \{1\}$, ρ is a congruence on A (or A^{*}) generated by R and $R \subseteq A^{+} \times A^{+}$ (or $R \subseteq A^{*} \times A^{*}$). As usual, we will use the notations $Sg(\pi)$ and $Mon(\pi)$ for the semigroup and the monoid presented by the presentation $\pi = \langle A|R\rangle$, respectively.
Through the paper p and q are odd primes, α,β,r and s are positive numbers such that $r,s \geq 2$. Without lose of generality suppose that $p^{\alpha} < q^{\beta}$ and consider the presentation $\pi_{k,t} = \langle a|a^{k+1} = at\rangle$

. Let $T_1 = Sg(\pi_p\alpha_{,r}) = \langle a\rangle, T_2 = Sg(\pi_q\alpha_{,s}) = \langle b\rangle$ and $S = T_1 \rtimes\phi\ T_2$. As a preliminary result on the semigroups we get:

Lemma 1.1. *For every positive integer k the relators $a^{kp\alpha} = a^{k(r-1)}$ and $a^{pk\alpha} = a^{(r-1)k}$ hold in the semigroup T_1. Moreover, the group End (T_1) possesses an involution element.*

Proof. Consider the relator $a^{p\alpha+1} = a^r$. Since $a^{p\alpha} = a^{p\alpha+1-1} = a^{r-1}$ then both of the- relators hold for $k = 1$. Now, by an induction method on k and using the induction hypothesis we get:

$$\begin{aligned} a^{(k+1)p^\alpha} &= a^{kp^\alpha}.a^{p^\alpha} \\ &= a^{k(r-1)}.a^{p^\alpha} \\ &= a^{k(r-1)}.a^{r-1} = a^{(k+1)(r-1)} \end{aligned}$$

and

$$\begin{aligned} a^{p^{(k+1)\alpha}} &= a^{p^{k\alpha+\alpha}} \\ &= (a^{p^{k\alpha}})^{p^\alpha} = (a^{(r-1)^k})^{p^\alpha} \\ &= (a^{p^\alpha})^{(r-1)^k} = (a^{r-1})^{(r-1)^k} = a^{(r-1)^{k+1}} \end{aligned}$$

To complete the proof we may defne the homomorphism $\theta \in End\ (T_1)$ by $\theta\ (a) = a^{p\alpha-r}$ Then,

$$\begin{aligned} \theta^2(a) = \theta(a^{p^\alpha-r}) &= a^{(p^\alpha-r)^2} = a^{p^\alpha(p^\alpha-2r)+r^2} \\ &= a^{(p^\alpha-2r)(r-1)+r^2} = a^{(r-1)p^\alpha-2r^2+2r+r^2} = a^{(r-1)^2-r^2+2r} = a \end{aligned}$$

$$\theta^3(a) = \theta(a) = a^{p^\alpha-r} = \theta(a).$$

Consequently, $\theta^3 = \theta$. □

By this endomorphism we may define the mapping $\phi : T_2 \rightarrow End(T_1)$ as follows.

$$\phi_{b^j}(a^i) = \begin{cases} \theta(a^i) & \text{if} j \text{ is odd,} \\ a^i, & \text{otherwise.} \end{cases}$$

for every values of i and j where, $1 \leq i \leq p^\alpha$ and $1 \leq j \leq q^\beta$. The equation $\phi(b^j b^{j'}) = \phi(b^j)\phi(b^{j'})$ may be proved by considering four possible cases for j and j'. Then, ϕ is a semigroup homomorphism.

This definition makes possible to formulate the multiplication on the semigroup $S = T_1 \rtimes\phi\ T_2$ as follows:

$$(a^i, b^j)(a^k, b^l) = \begin{cases} (a^{i+(p^\alpha-r)k}, b^{j+l}) & \text{if} j \text{ is odd,} \\ (a^{i+k}, b^{j+l}), & \text{otherwise.} \end{cases}$$

Every element of S may be presented explicitly in terms of the elements
$x = (a,b)$, $Ai = (a^i,b)$, $B_j = (a,b^j)$, $(i = 2,3,\dots,p^\alpha)$, $(j = 2,3,\dots,q^\beta)$
Indeed,

Lemma 1.2. *If* $X = \{x, A_i, B_i | 2 \leq i \leq p^\alpha, 2 \leq j \leq q^\beta\}$ *then X generates S.*

Proof. It is suffcient to show that every element (a^i,b^j) may be rewritten as a product of the elements of X, for every i and j when $2 \leq i \leq\alpha\ p^\alpha$ and $2 \leq j \leq\beta\ q^\beta$. Indeed, by the defined multiplication on S and by considering the relators $a^{p+1} = a^r$ and $b^{q+1} = b^s$ of the semigroups T_1 and T_2, we get:

$$A_{i+1}B_{j-1} = (a^{i+1}, b)(a, b^{j-1}) = (a^{i+1+(p^\alpha - r)}, b^j) = (a^{p^\alpha+1+i-r}, b^j) = (a^i, b^j).$$

Also, the following key lemma gives us useful information of the powers of elements of the semigroup S. These information could be applicable in study of the power graph of S. □

Lemma 1.3. *(i). For every i and k where* $2 \le i \le p^\alpha$ *and* $1 \le k \le \frac{q^{\beta-1}}{2}$,

$A_i^{2k} = (a^{p^\alpha - r+1}, b^{2k})$, *and* $A_i^{2k+1} = (a^i, b^{2k+1})$,

(ii). for every i and k where $2 \le i \le p^\alpha$ *and* $1 \le k \le \frac{q^{\beta-1}}{2}$

The powers are reduced modulo p^α- 1.
(iii). For every i where $2 \le i \le q^\beta$.

$$x^i = \begin{cases} (a^{p^\alpha - r+1}, b^i), & \textit{if } i \textit{ is even,} \\ (a, b^i), & \textit{if } i \textit{ is odd.} \end{cases}$$

Proof. Proofs are easy by using induction methods and considering the results of Lemma 1.2.

2 The power graphs

The semigroups $T1$; $T2$ and S are as in the last section. First of all, we follow [5, 6] and recall two
results on the power graphs of the abelian groups.

Lemma 2.1. *(Theorem 2,12 of [5]). For a finite group G,* $P(G)$ *is complete if and only if G is the cyclic group of order of a prime.*

Lemma 2.2. *(Theorem 5 of [6]). G is a finite group of order* p_1q_1 *where* p_1 *and* q_1 *are primes and* $p_1 > q_1$ *. Then,*

(i). G is cyclic if and only if $P(G) \simeq (K_{p1}-1 \cup K_{q1}-1) + K_{\phi(p1q1)+1}$, *($\phi$ is the well-known Eulerian function).*

(ii). G is not cyclic if and only if $P(G) \simeq K_1 + (pK_{q1}-1 \cup K_{p1}-1)$.

Proposition 1. *For every positive integer α and an odd prime p let* T_1 *be the non- group monogenic semigroup presented by the presentation* $\langle a | a^{p\alpha+1=a2}\rangle$. *Then,* $P(T_1) \simeq K_p\alpha$ *if and only if* $p^\alpha - 1$ *is a power of a prime or is a product of two different primes.*

Proof. Suppose that the graph $P(T)$ is a complete and n is not a power of a prime. Then there exist at least two different primes p_1 and p_2 dividing n. So, c^{p1} and c^{p2} as the vertices of $P(T)$ are adjacent. This means that for some positive integers k and k',$c^{p2} = c^{kp1}$ or $c^{p1} = c^{k'p2}$. By using he results of Lemma 1.1 the relator $a^{n+1} = a^t$ yields the relator $a^{k(n+1-t)} = a^{n+1-t}$, for every integer $k \ge 2$. Now, in the case when $c^{p2} = c^{kp1}$ we get the equation $p_2 = p_1+(n-t+1)$. Hence, p_2 divides $p_1 - t + 1$, i.e.; $p_1 - t + 1 = t_1p_2$, for some positive integer t_1. Eliminating p_1 in

$$\begin{cases} p_2 = p_1 + n - t + 1 \\ p_1 - t + 1 = t_1 p_2 \end{cases}$$

Gives us the contradiction $p_2 = n + t_1p_2 > p_2$. A similar contradiction occurs when $c^{p1} = c^{k'p2}$.

Consequently, n is as a power of a prime. Conversely, every element $a^i,(i \geq 2)$ of the semigroup $T_1 = \{a,a^2,...,a^p\alpha\}$ as a vertex of $P(T_1)$ is adjacent with the vertexα a. Moreover, $G = \{a^2,...,a^{p\alpha}\}$ is a cyclic subgroup of T_1 with the identity element $e = a^{p-1}$. This group may be generated by $c = a^{p\alpha}$, for,
$c^i = a^{ip^\alpha} = a^{i(2-1)} = a^i$,(by Lemma1.1and setting $r = 2$)
For every $i = 2,3,...,p^\alpha-1$. Suppose that $p^\alpha-1$ is a power of a prime p_0, since p is odd then $p_0 = 2$. Hence, by the Lemma 2.1, $P(G)$ is complete and then $P(T_1)$ is so.

In the case when $p^\alpha - 1$ is a product of two different primes, a same proof may be given by by using the Lemma 2.2, because G is a cyclic group. □

In this proposition we studied the power graph of the semigroup T_1 when $r = 2$. When $r \geq 3$, the power graph of the corresponding subgroup may or may not be cyclic. Hence, the above lemmas on the power graphs of finite groups are not applicable to study of the power graphs of semigroups. In the following proposition we study the case $r \geq 3$ and show that the abelianity of the corresponding subgroup may cause the Eulerianity of the power graph of the semigroup T_1.

Proposition 2. *For every positive integers $\alpha,r \geq 2$ and any odd prime p, the semigroup T_1 contains a cyclic subgroup G_1 of order $p^\alpha - r + 1$. Moreover, if $P(G_1)$ is complete then $P(T_1)$ is Eulerian.*

Proof. As well as in the last proposition, each element of the subset $\{a^2,...,a^\alpha\}$ of T_1 is a power of the element $a \in T_1$. Then, the vertex a is adjacent with all other vertices of $P(T_1)$. Since $a^p\alpha^{+1} = a^r$ then T_1 contains the cyclic subgroup

$$G_1 = \{a^r, a^{r+1}, ..., a^{p^\alpha}\}$$

of order $a^{p\alpha}-r+1$. This may be proved by considering the elements $a_1 = a^{p\alpha-r+2}$ and $c_1 = a^{p\alpha-r+1}$. The element a1 generates G_1, for,

$$a_1^i = a^{ip^\alpha-ir+2i} = a^{i(r-1)}.a^{2i-ir} = a^{ir-i+2i-ir} = a^i$$

Where, $r \leq i \leq p^\alpha$. And c_1 is the identity element of G_1 because of the following relators:

$$c_1a^i = a^{p^\alpha-r+1+i} = a^{p^\alpha+1}a^{i-r} = a^ra^{i-r} = a^i \qquad r \leq i \leq p^\alpha.$$

To complete the proof suppose that the graph $P(G_1)$ is complete. Then, any two vertices of$P(G_1)$ are adjacent. By considering the vertices $\{a^2,...,a^{r-1}\}$ of the graph $P(T_1)$ we have to show that at least one vertex a^i of this set is adjacent with at least one vertex of $P(G_1)$. Consider two cases for r. For even values of r, consider the vertex a^2 where we get $(a^2)^{\frac{r}{2}} = a^r$ So, (a^2) and (a^r) are adjacent in this case, and (a^2) is adjacent with a^{r+1} when r is odd, i.e. $(a^2)^{\frac{r+1}{2}} = a^{r+1}$ Consequently, the completeness of the graph $P(G_1)$ yields that $P(T_1)$ is Eulerian. □

To study the power graph of the semigroup S recall the parameters p,q,r and s as well as in Section 1 where we set $r = s = 2$. Our result concerning the graph $P(S)$ is:

Proposition 3. *The semigroup S possesses a unique non-abelian maximal sub group G of order* $(p^{\alpha} - r + 1)(q^{\beta} - s + 1)$. *Moreover, if* $P(G)$ *is complete then* $P(S)$ *is Eulerian.*

Proof. We construct the group $G = G_1 \rtimes G_2$ such that $G_1 = \langle a_1 \rangle$ and $G_2 = \langle a_2 \rangle$ where, $a_1 = a^{p\alpha - r+1}$ and $a_2 = b^{q}\beta^{-s+1}$. Note that the semidirect product of a group by another group is defined as similar as in the semigroups except when *End* will be changed to *Aut.* As well as in the last proposition it may be proved that $c_1 = a^{p\alpha - r+1}$ and $c_2 = b^{q\beta - s+1}$ are the identity elements of the groups G_1 and G_2, respectively. Evidently, by letting $r = s = 2$, G is the unique maximal subgroup of the semigroup S and $S = X \cup G$. Suppose that the graph $P(G)$ is complete then, to prove that $P(S)$ is Eulerian it is sufficient to show that every element of X (as a vertex of $P(S)$) is adjacent with at least one vertex of $P(G)$ Indeed, Lemma 1:3-(i) yields,

$$A_i^2 = (a^{p^{\alpha} - r + 1}, b^2), \quad (i = 2, 3, ..., p^{\alpha}).$$

So, each A_i adjacent with $(a^{p\alpha - r+1, b2)}$. Also, the part (ii) of the same lemma gives us

3 Conclusion

The corresponding maximal subgroup G of the semigroup S where $r = s = 2$, is a non-abelian group of order $(p^{\alpha} - 1)(q^{\beta} - 1)$. During the following examples we examine certain subclasses of S to determine related behaviours of the power graphs of S and G.

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