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GRAPH'S MONOPHONIC VERTEX COVERING NUMBER

M. Helen Sheeba[1], K. A. Francis Jude Shini[2], S. Durai Raj[3]
[1]Assistant Professor, Department of Mathematics, Pioneer Kumaraswamy College, Nagercoil – 629 003, Tamil Nadu, India. Email: hsheeba67@gmail.com
[2]Research Scholar, Reg No: 20213132092001, Department of Mathematics, Pioneer Kumaraswamy College, Nagercoil – 629 003, Tamil Nadu, India. Email: shinishini111@gmail.com
[3]Associate Professor and Principal(Rtd), Department of Mathematics, Pioneer Kumaraswamy College, Nagercoil – 629 003, Tamil Nadu, India. Email: durairajsprincpkc@gmail.com
Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli – 627 012, Tamil Nadu, India.

Abstract
For a connected graph G of order $n \geq 2$, a set S of vertices of G, is monophonic vertex cover of G if S is both a monophonic set and a vertex cover of G. The minimum cardinality of a monophonic vertex cover of G is called the monophonic vertex covering number of G and is denoted by m_α (G). Any monophonic vertex cover of cardinality m_α (G) is a m_α-set of G. Some general properties satisfied by monophonic vertex cover are studied. The monophonic vertex covering number of several classes of graphs are determined.
Keywords: monophonic set, vertex covering set, monophonic vertex cover, monophonic vertex covering number.

1. Introduction

By a graph G = (V, E), we mean a finite undirected simple connected graph. The order and size of G are denoted by n and m respectively. For basic graph theoretic terminology we refer to Harary[12]. The distance *d(u,v)* between two vertices u and v in a connected graph G is the length of a shortest u-v path in G[4]. For a vertex v of G, the eccentricity $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the radius, *rad G* and the maximum eccentricity is its diameter, *diam G*. The neighbourhood of a vertex v of G is the set $N(v)$ consisting of all vertices which are adjacent with v. A vertex v is a simplical vertex or an extreme vertex of G if the subgraph induced by its neighbourhood $N(v)$ is complete. A caterpillar is a tree of order 3 or more, the removal of whose end vertices produces a path called the spine of the caterpillar. A diametral path of a graph is a shortest path whose length is equal to the diameter of the graph. A tree containing exactly two non-pendent vertices is called a double star denoted by S_{k_1,k_2} where k_1 and k_2 are the number of pendent vertices on these two non-pendent vertices. A graph G is called triangle free if it does not contain cycles of length 3. A set of vertices no two of which are adjacent is called an independent set. By a matching in a graph G, we mean an independent set of edges of G. A maximal matching is a matching M of a graph G that is not a subset of any other matching. The independence number $\beta(G)$ of G is the maximum number of vertices in an independent set of vertices of G. A subset $S \subseteq V(G)$ is a dominating set if every vertex in $V-S$ is adjacent to at least one vertex in S. A set $S \subseteq V(G)$ is called a global dominating set if it is a dominating set of both G and $\bar{G}$ (the complement of G). The minimum cardinality of a dominating set in a graph G is called the dominating number of G and denoted by $\gamma(G)$. The dominating number is further studied in [1-3,10-11].

A geodetic set of G is a set S⊆V(G) such that every vertex of G is contained in a geodesic joining some pair of vertices in S. The geodetic number $g(G)$ of G is the minimum cardinality of its geodetic sets. The geodetic number of a graph was introduced in [6] and further studied in [5,7]. A subset $S \subseteq V(G)$ is called geodetic global dominating set of G if S is both geodetic and global dominating set of G. The geodetic global domination number of a graph was introduced in [15] and further studied in [16,17]. A chord of a path P is an edge joining two non-adjacent vertices of P. A path P is called a monophonic path if it is a chordless path. A set S of vertices of G is a monophonic set of G if each vertex v of G lies on an x-y monophonic path for some $x,y \in S$. The minimum cardinality of a monophonic set of G is the monophonic number of G and is denoted by m(G). Any monophonic set of cardinality m(G) is a minimum monophonic set or a monophonic basis or a $m-set$ of G. The monophonic number of a graph was studied in [8,9] and discussed in [13,18]. A subset $S \subseteq V(G)$ is said to be a vertex covering set of G if every edge has at least one end vertex in S. A vertex covering set of G with the minimum cardinality is called a minimum vertex covering set of G. The vertex covering number of G is the cardinality of any minimum vertex covering set of G. It is denoted by $\alpha(G)$ [19]. A set of vertices of G is said to be monophonic domination set if it is both a monophonic set and a dominating set of G. The minimum cardinality of a monophonic domination set of G is called a monophonic domination number of G and denoted by $\gamma_m(G)$. The monophonic domination number was studied in [14].

The following theorems will be used in the sequel.

Theorem 1.1.[18] Every extreme vertex of a connected graph G belongs to every monophonic set of G. In particular, each end vertex of G belongs to every monophonic set of G.

Theorem1.2.[18] For any tree T with k end vertices, $m(T)=k$. In fact, the set of all end vertices of T is the unique monophonic set of T.

Throughout this paper G denotes a connected graph with at least two vertices.

2. MONOPHONIC VERTEX COVER

Definition2.1. Let G be a connected graph of order $n \geq 2$. A set S of vertices of G is a monophonic vertex cover of G if S is both a monophonic set and a vertex cover of G. The minimum cardinality of a monophonic vertex cover of G is called the monophonic vertex covering number of G and is denoted by m_α(G). Any monophonic vertex cover of cardinality m_α (G) is a m_α-set of G.

Example2.2. For the graph G given in Figure 2.1, S = $\{v_1, v_5\}$ is a minimum monophonic set of G so that m(G) = 2 and S' = $\{v_1, v_4, v_5\}$ is a minimum monophonic vertex cover of G so that

m_α (G)=3. Thus the monophonic number is different from the monophonic vertex covering number of a graph G.

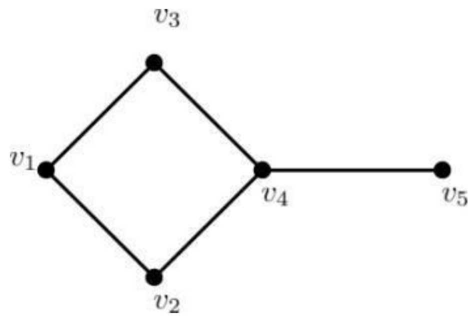

Figure 2.1 G

Remark 2.3. For the graph G given in Figure 2.2, $S=\{v_2, v_3\}$ is a minimum monophonic set of G so that $m(G) = 2$. S is also a minimum monophonic dominating set of G so that γ_m(G)=2. $S'=\{v_1, v_2, v_3\}$ is a minimum monophonic vertex cover of G so that m_α (G)=3. Hence the monophonic vertex covering number of a graph is different from the monophonic number and monophonic dominating number of a graph G.

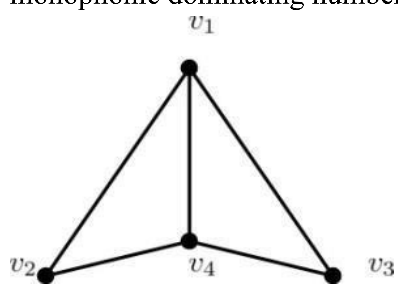

Figure 2.2 G

Theorem 2.4. For any connected graph G, $2 \leq max\ \{\alpha(G), m(G)\} \leq m_\alpha\ (G) \leq n$.

Proof of theorem 2.4. Any monophonic set of G needs at least 2 vertices. Then $2 \leq max\{\alpha(G), m(G)\}$. From the definition of monophonic vertex cover of G, we have, $max\ \{\alpha(G), m(G)\} \leq m_\alpha\ (G)$. Clearly V($G$) is a monophonic vertex cover of G. Hence $m_\alpha\ (G) \leq n$. Thus $2 \leq max\ \{\alpha(G), m\ (G)\} \leq m_\alpha\ (G) \leq n$.

Remark 2.5. The bounds in Theorem 2.4 are sharp. For the complete graph K_4, $m_\alpha\ (K_4) = 4$. The bounds are strict in Figure 2.3 as $\alpha(G)$=2, $m(G)$=3, m_α (G)=4. Here 2<3<4<5.

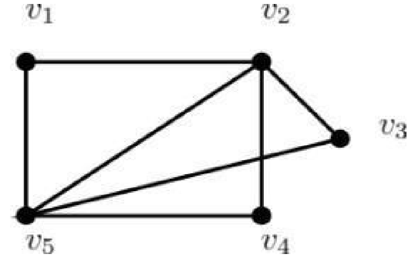

Figure 2.3 G

Remark 2.6. Clearly union of a vertex covering set and a monophonic set of G is a monophonic vertex cover of G. In Figure 2.1, $S = \{v_1, v_4, v_5\}$ is a monophonic vertex cover and in Figure 2.2, S= $\{v_1, v_2, v_3, v_4\}$ is a monophonic vertex cover.

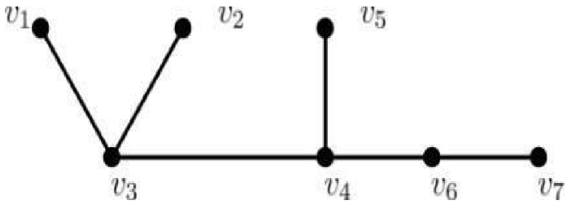

Figure 2.4 G

Thus 2 ≤ *max* $\{\alpha(G), m(G)\} \leq m_\alpha(G) \leq$ *min* $\{\alpha(G)+m(G), n\}$.

For the graph G in Figure 2.4, we observe that $S_1 = \{v_3, v_5, v_6\}$ is a minimum vertex cover of G so that $\alpha(G) = 3$, $S_2 = \{v_1, v_2, v_5, v_7\}$ is a minimum monophonic set of G so that m(G)=4 and $S_3 = \{v_1, v_2, v_3, v_5, v_6, v_7\} = S_1 \cup S_2$ is a m_α-set of G and so $m_\alpha(G) = 6 < n = 7$.

Theorem 2.7. Each extreme vertex of G belongs to every monophonic vertex cover of G. In particular, each end vertex of G belongs to every monophonic vertex cover of G.

Proof of theorem 2.7. From the definition of m_α-set, every m_α-set of G is a m-set of G. Hence the result follows from Theorem 1.1.

Corollary2.8. For any graph G with k extreme vertices, $max\{2, k\} \leq m_\alpha(G) \leq n$.

Proof of corollary 2.8. The result follows from Theorem 2.4 and Theorem 2.7.

Corollary 2.9. Let $K_{1,n-1}$(n≥3) be a star. Then $m_\alpha(K_{1,n-1}) = n - 1$.

Proof of corollary 2.9. Let x be the centre and S = $\{v_1, v_2, \ldots, v_{n-1}\}$ be the set of all extreme vertices of $K_{1,n-1}$(n ≥ 3). Clearly S is a minimum monophonic vertex cover of $K_{1,n-1}$(n≥3) by Theorem 2.7. Hence $m_\alpha(K_{1,n-1}) = n-1$.

Corollary2.10. For the complete graph K_n(n≥2), $m_\alpha(K_n)$=n.

Proof of corollary 2.10. We have every vertex of the complete graph $K_n(n \geq 2)$ is an extreme vertex. Then by Theorem 2.7, the vertex set is the unique monophonic vertex cover of K_n. Then $m_\alpha(K_n) = n$.

Theorem2.11. If G is a connected graph of order $n \geq 2$, then

(i) $m_\alpha(G) = 2$ if and only if G is either K_2 or $K_{2,n-2}(n \geq 3)$.

(ii) $m_\alpha(G) = n$ if and only if G = $K_n(n \geq 2)$.

Proof of theorem 2.11.

(i) Let m_α (G)=2. Let S = {u,v} be a minimum monophonic vertex cover of G. We claim that G = K_2 or $K_{2,n-2}$ ($n \geq 3$). Suppose that G = K_2. Then there is nothing to prove. If not, then $n \geq 3$ and since $S = \{u, v\}$ is a m_α -set of G, u and v cannot be adjacent in G. Let $W= V - S$. We claim that every vertex of W is adjacent to both u and v and no two vertices of W are adjacent. Suppose there is a vertex $w \in W$ such that w is adjacent to at most one vertex in S. Then w lies on a u-v monophonic path of length at least 3. Let P: $u = v_0, v_1, v_2, \dots, v_i = w, v_{i+1}, \dots, v_m = v$ be a u-v monophonic. Then the edges in E(P)- $\{v_0v_1, v_{m-1}v_m\}$ are not covered by any of the vertices u and v, which is a contradiction to S is a m_α -set. Hence every vertex of W is adjacent to both u and v. Suppose there exist vertices $w_i, w_j \in W$ such that w_i and w_j are adjacent. Since every vertex of W is adjacent to both u and v and $S = \{u,v\}$ is a m_α-set of G, w_i and w_j lie on the u-v monophonic paths $uw_i\, v$ and $uw_j\, v$ respectively. Then the edge w_iw_j is not covered by any of vertices of S, which is a contradiction to S is a m_α - set of G. Hence no two vertices of W are adjacent in G. Thus G is the complete bi partite graph $K_{2,n-2}$ ($n \geq 3$) with the partite sets S and W.

Conversely assume that G = K_2 or $K_{2,n-2}$ ($n \geq 3$). If G = K_2, then by Corollary 2.10, m_α (K_2) = 2. If not, let G = $K_{2,n-2}$ ($n \geq 3$). Let U = $\{u_1u_2\}$ and W = $\{w_1, w_2, \dots, w_{n-2}\}$ be the bipartition of G. Clearly every vertex $w_i(1 \leq i \leq n-2)$ lies on the monophonic path $u_1w_iu_2$ and the vertices u_1 and u_2 cover all the edges of G. Hence U is a monophonic vertex cover of G and so m_α (G)=2.

(ii) Assume that G = K_n ($n \geq 2$). Then by Corollary 2.10, m_α $(G) = n$. Conversely assume that $m_\alpha(G) = n$. We claim that G = K_n ($n \geq 2$). For n=2, the result holds from (i). Let $n \geq 3$. Suppose there exist two non-adjacent vertices u and v in G. Let a vertex x be adjacent to u lying on a u-v monophonic. Then $V(G) - \{x\}$ is a monophonic vertex cover of G, which is a contradiction to $m_\alpha(G) = n$. Thus G=K_n.

Theorem.2.12. For a connected graph G with $m(G) \geq n-1$, m_α $(G) = m(G)$.

Proof of theorem 2.12. Let G be a connected graph with $m(G) \geq n - 1$. Then by Theorem 2.4, $m(G) \leq m_\alpha(G) \leq n$. Now, if $m(G) = n$, then $m_\alpha(G) = n$. Hence m_α $(G) = m(G)$. If $m(G) = n - 1$, then let S = $\{x_1, x_2, \dots, x_{n-1}\}$ be a minimum monophonic set of G. Let $x \notin S$ be a vertex of G. Then any edge xx_i $(1 \leq i \leq n-1)$ lies on a monophonic path joining pair of vertices of S and every edge of G has at least one end point in S. Hence S is a minimum monophonic vertex cover of G and so m_α $(G) = m(G)$.

Remark.2.13. The converse of Theorem 2.12 need not be true. For the graph in Figure 2.5, $S = \{v_1, v_2\}$ is both a m-set of G and a m_α-set of G. Hence m_α (G)= $m(G)$ =2 but $m(G) < n-1$.

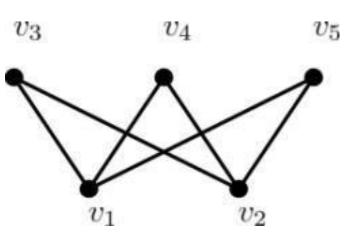

Figure 2.5 G

Theorem.2.14. For a connected graph G of order $n \geq 2$, $m_\alpha(G) = m(G)$ if and only if there exists a minimum monophonic set of G such that $V(G) - S$ is either empty or an independent set.

Proof of theorem 2.14. Assume that $m_\alpha(G) = m(G)$. Let S = $\{v_1, v_2, \dots, v_k\}$ be a minimum monophonic vertex cover of G. Then S is also a minimum monophonic set of G. If n=k, then V(G)– S is empty. Let $n > k$. If not, there exist two vertices $u, v \in V(G) - S$ such that $uv \in E(G)$. Then the edge uv has none of its end vertices in S, which is a contradiction. Hence there exists a minimum monophonic set of G such that V(G)-S is either empty or an independent set.

Conversely assume that there exists a minimum monophonic set of G such that V(G)– S is either empty or an independent set. Let S = $\{v_1, v_2, \dots, v_k\}$ so that m(G)=|S|. Suppose V(G)– S is empty. Then n=k and S=V(G). Hence S is a minimum monophonic vertex cover of G so that m_α(G) = m(G). If not, let V(G)– S be independent. Then every edge of G has at least one end in V(G)−(V(G)−S) = S and so S is a vertex cover of G. Thus S is a minimum monophonic vertex cover of G. Thus m_α(G) = m(G).

Theorem.2.15. For the cycle $C_n (n \geq 4)$, $m_\alpha(C_n) = \left\lceil \frac{n}{2} \right\rceil$.

Proof of theorem 2.15. Let C_n: $v_1\ v_2\ \dots\ v_n\ v_1$ be a cycle of order n. Here $S = \{v_1, v_3, v_5, \dots, v_{2\lceil \frac{n}{2} \rceil - 1}\}$ is a minimum monophonic vertex cover of C_n. Hence $m_\alpha(C_n) = \left\lceil \frac{n}{2} \right\rceil$.

Theorem.2.16. Let T be a tree of order $n \geq 2$. Then the following statements are equivalent.

(1) $m_\alpha(T) = m(T)$.

(2) T is a star.

(3) $\alpha(T)=1$.

(4) The set of all end vertices of T is a vertex cover of T.

Proof of theorem 2.16. Let S be the set of all end vertices of T. Since T is a tree, from the Theorem 1. 2, we have, S is the unique m-set of T.

(1) ⇒ (2) Assume that $m_\alpha(T) = m(T)$.We claim that T is a star. If not, then $diam\ T \geq 3$. Then T has at least one edge other than the end edges. Let S' be the set of all edges of T which are not end edges. Then clearly no edges of S' have its end vertices in S. Hence S is not a vertex cover of T. By Theorem 2.7, any monophonic vertex cover of T contains S. Hence $m_\alpha(T) > |S| = m(T)$, which is a contradiction to $m_\alpha(T)=m(T)$.

(2) ⇒(3) Assume that T is a star. If n=2, then an end vertex of T will cover the edge of T. If $n \geq 3$, then the cut vertex of T will cover all the edges in T. Hence $\alpha(T)=1$.

(3) ⇒(4)Assume that $\alpha(T)=1$.Then there exists a vertex say x in T such that x is an end vertex of all the edges in T. Hence all the edges in T are the end edges in T and so S forms a vertex cover of T.

(4)⇒(1) Assume that S is a vertex cover of T. Then by Theorem 1.2, S is a m-set of T and by Theorem 2.7, S is a m_α-set of T. Hence m_α (T)=$m(T)$.

Remark 2.17. The results in Theorem 2.16 are not equivalent for any connected graph G of order $n \geq 2$. For the graph G in Figure 2.6, S= $\{v_1, v_2, v_3\}$ is both m-set andm_α-set of G. So $m_\alpha(G)$ = $m(G)$ = 3. Also, S is a minimum vertex covering set and so $\alpha(G)$=3. And here G is not a star.

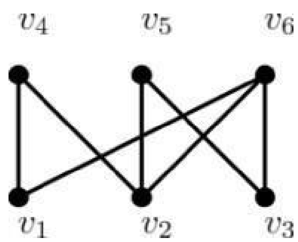

Figure 2.6 G

3. Conclusion

In this paper we analyzed the monophonic vertex covering number of a graph. It is more interesting to continue my research in this area and it is very useful for further research.

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