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CONCEPTION OF POSITIVE INTEGER SOLUTIONS RELATING JACOBSTHAL AND JACOBSTHAL – LUCAS NUMBERS TO RESTRICTED NUMBER OF QUADRATIC EQUATIONS WITH DOUBLE VARIABLES

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ABSTRACT:
This study unveils patterns of positive integer solutions for limited number of explicit binary quadratic equations encompassing Jacobsthal and Jacobsthal-Lucas numbers by means of the pertinent features connecting these two numbers and the notions of divisibility.
KEYWORDS: quadratic Diophantine equations, Jacobsthal and Jacobsthal-Lucas numbers

INTRODUCTION:

In [7, 8, 9], the authors scrutinized various Diophantine equations using generalized Fibonacci and Lucas sequences. In [9], the writers premeditated the specific Diophantine equation $x^2 - kxy + y^2 + x = 0$. In [19], Pingzhi Yuan, Yongzhong discoursed on the quadratic Diophantine equation with two variables $x^2 - kxy + y^2 + lx = 0$, when $l \in \{1, 2, 4\}$. [1-6,10-18] may be referred for a comprehensive evaluation. In this communication, sequences of non-negative integer solutions for restricted number of quadratic equations with double variables $X^2 - XY - 2Y^2 = \pm C, X^2 - 5XY \pm 4Y^2 = \pm C, X^2 - XY - 2Y^2 \pm CX = 0, X^2 - XY - 2Y^2 \pm CY = 0, X^2 - 5XY - 4Y^2 \pm CX = 0, X^2 - 5XY - 4Y^2 \pm CY = 0$, $X^2 - XY - 2Y^2 = \pm 9C$, $X^2 - XY - 2Y^2 \pm 9CX = 0$, $X^2 - XY - 2Y^2 \pm 9CY = 0$, $X^2 - 5XY + 4Y^2 = \pm 9C$, $X^2 - 5XY + 4Y^2 \pm 9CX = 0$ and $X^2 - 5XY + 4Y^2 \pm 9CY = 0$ where C is a fixed constant which is some powers of the number 2 encircling Jacobsthal and Jacobsthal-Lucas numbers are investigated by utilizing the appropriate erections connecting these two numbers and the concepts of divisibility.

Needed Theorems:

Theorem: [I]

For each integer $n > 1$, there exists primes $p_1 \leq p_2 \leq \cdots \leq p_r$ such that $n = p_1 p_2 \ldots p_r$,this factorization is unique.

Theorem: [II]

If positive integers x, y, a, b, c with $gcd(x, c) = 1$ satisfying the equations

$$x^2 - axy - by^2 \pm cx = 0$$

then $x = u^2$ and $y = uv$ for some positive integers u and v.

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PRIMARY CONSEQUENCES:

The n^{th} Jacobsthal number designated by J_n is delineated by $J_n = J_{n-1} + 2J_{n-2}$, for $n \geq 2$ where $J_0 = 0, J_1 = 1$. If α, β are two roots of the equation $x^2 - x - 2 = 0$, then $\alpha = -1, \beta = 2$ such that $\alpha\beta = -2$ and $\alpha + \beta = 2$. Furthermore, it is well- recognized and modest to reveal the characteristics that $\alpha^{n-1} = J_n - \beta J_{n-1}$ and $\beta^{n-1} = J_n - \alpha J_{n-1}$ for every $n \in \mathbb{Z}$, the set of all integers. Also, it might be declared by Mathematical induction that $J_n{}^2 - J_n J_{n-1} - 2J_{n-1}{}^2 = (-2)^{n-1}$ $\forall n \in \mathbb{Z}$. (1)

Similarly, the n^{th} Jacobsthal-Lucas number j_n is described as $j_n = j_{n-1} + 2j_{n-2}$ for $n \geq 2$ and

$j_0 = 2, j_1 = 1$. The interrelation between Jacobsthal and Jacobsthal-Lucas numbers are approved as follows

1. $j_n = J_{n+1} + 2J_{n-1}$ for every $n \in \mathbb{Z}$.
2. $j_n{}^2 - j_n j_{n-1} - 2j_{n-1}{}^2 = -9(-2)^{n-1}$ for every $n \in \mathbb{Z}$ (2)

Theorem: 1

The constitutive criterion for all non-negative integer solutions to the specific second-degree equation involving two variables $X^2 - XY - 2Y^2 = C(-1)^{n-1}$ is $(X, Y, C) = (J_n, J_{n-1}, 2^{n-1})$ with $n \geq 1$.

Proof:

If $(X, Y, C) = (J_n, J_{n-1}, 2^{n-1})$, then it follows from identity (1) that $X^2 - 2XY - Y^2 = C(-1)^{n-1}$. Conversely suppose that $X^2 - XY - 2Y^2 = C(-1)^{n-1}$ for some positive integers X, Y and $C = 2^{n-1}$.

Then, $(X - \alpha Y)(X - \beta Y) = (\alpha\beta)^{n-1} \Rightarrow (X - \alpha Y)(X - \beta Y) = (J_n - \beta J_{n-1})(J_n - \alpha J_{n-1})$.

Thus, $X - \alpha Y = J_n - \alpha J_{n-1}$ and hence $(X, Y, C) = (J_n, J_{n-1}, 2^{n-1}), n \geq 1$.

Corollary: 1.1

The viable solutions to the certain quadratic equation $X^2 - XY - 2Y^2 = C$ are enumerated by $(X, Y, C) = (J_{2m+1}, J_{2m}, 2^{2m})$, $m \geq 0$.

Corollary: 1.2

Every conceivable solution in Jacobsthal numbers of the equation $X^2 - XY - 2Y^2 = -C$ are quantified by $(X, Y, C) = (J_{2m}, J_{2m-1}, 2^{2m-1})$ with $m \geq 1$.

Theorem: 2

The trustworthy integer solutions to the exact equation $X^2 - 5XY + 4Y^2 = C$ are conquered by $(X, Y, C) = (J_{2n+2}, J_{2n}, 2^{2n})$ with $n \geq 0$.

Proof:

For our convenience, let us choose $X > 2Y$

Then, $X^2 - 5XY + 4Y^2 = C \Rightarrow (X - 2Y)^2 - (X - 2Y)Y - Y^2 = C$

By corollary 1.1, it should be $X - 2Y = J_{2n+1}$, $Y = J_{2n}$ and $C = 2^{2n}$

The first two of the above equations yields the value of X as $X = J_{2n+2}$

Hence, the solutions to the required equation are mentioned by $(X, Y, C) = (J_{2n+2}, J_{2n}, 2^{2n}), n \geq 0$.

Corollary: 2.1

The infinitely many positive integer solutions to the equation $X^2 - 5XY + 4Y^2 = -C$ are attained by $(X, Y, C) = (J_{2n+1}, J_{2n-1}, 2^{2n-1})$ with $n \geq 1$.

Theorem: 3

Let X, Y be any two natural numbers sustaining the equation $X^2 - XY - 2Y^2 \pm CX = 0$, then $X = U^2$ and $Y = UV$ where $U, V \in \mathbb{N}$.

Proof:

Modify the original equation as $X(X - Y \pm C) = 2Y^2$

It is easy to see that X/Y^2 and hence $Y^2 = XZ$ for some natural number Z.

If p is any prime number such that $p|X$ and $p|Z$, then $p|Y$

This affords the expression $X - Y - 2Z \pm C = 0$ which guarantees that $p|C$.

Here, the only possible value of p is $p = 2$ which implies that $X = 2X_1, Y = 2Y_1$

Again, it grasps that ${X_1}^2 - X_1Y_1 - 2{Y_1}^2 \pm C_1X_1 = 0$ where $C_1 = C/2$.

Enduring the same method as enlightened above till the constant C vanishes, it is found that ${X_n}^2 - X_nY_n - 2{Y_n}^2 \pm X_n = 0$

It follows that $X_n|{Y_n}^2$ and hence ${Y_n}^2 = X_nZ_n$ for some positive integer Z_n

If a prime number p satisfying the conditions $p|X_n$ and $p|Z_n$, then $p|Y_n$

Then, it is detected that $X_n - Y_n - 2Z_n \pm 1 = 0$.

This equation infers that $p|1$ which is not possible.

Therefore, $gcd(X, Z) = 1$.

By the needed theorem [I] stated above, it is noted that $X = U^2$ and $Z = V^2$ for some positive integers U and V where $\gcd(U, V) = 1$.

Hence, it is concluded that $Y^2 = XZ = U^2V^2 \Rightarrow Y = UV$.

Corollary: 3.1

The probable values of X, Y in the equation $X^2 - XY - 2Y^2 + CX = 0$ are given by $(X, Y, C) = \left({J_{2n}}^2, J_{2n}\, J_{2n-1}, 2^{2n-1}\right), n \geq 1$.

Corollary: 3.2

The realistic solutions in Jacobsthal numbers to the equation $X^2 - XY - 2Y^2 - CX = 0$ are computed by $(X, Y, C) = \left({J_{2n+1}}^2, J_{2n+1}\, J_{2n}, 2^{2n}\right), n \geq 0$.

Theorem: 4

If X, Y be any two positive integers such that $X^2 - XY - 2Y^2 \pm CY = 0$, then $X = UV$ and $Y = U^2$ for some positive integers U and V with $\gcd(U, V) = 1$.

Proof:

The proof is analogous to Theorem 3.

Corollary: 4.1

The convincing integer values of X, Y in the equation $X^2 - XY - 2Y^2 + CY = 0$ are resolved by

$(X, Y, C) = \left(J_{2n}\, J_{2n-1}, {J_{2n-1}}^2, 2^{2n-1}\right), n \geq 1$.

Corollary: 4.2

The conventional solutions to the quadratic equation $X^2 - XY - 2Y^2 - CY = 0$ are particularized by $(X, Y, C) = \left(J_{2n+1} J_{2n}, {J_{2n}}^2, 2^{2n}\right), n \geq 0$.

Theorem: 5

The patterns of non-negative integer solutions to the equation $X^2 - 5XY + 4Y^2 + CX = 0$ are exemplified by $(X, Y, C) = \left({J_{2n+1}}^2, J_{2n-1}J_{2n+1}, 2^{2n-1}\right)$ where $n \geq 1$.

Proof:

Let X, Y be two non-negative integers such that $X^2 - 5XY + 4Y^2 + CX = 0$.

The alteration of the above equation $4Y^2 = X(5Y - X - C)$ ensures that X divides $Y^2$and henceforth $Y^2 = XZ$ for some non-negative integer Z.

Suppose that a certain prime number p divides both X and Z.

Then $p|Y$ and also the relation $X - 6Y + 4Z + C = 0$ holds for all $X, Y \in \mathbb{Z}^+$, the set of all positive integers.

Thus, p divides C and the chance of such p is $p = 2$.

This condition confirms that $X = 2X_1, Y = 2Y_1$ for some $X_1, Y_1 \in \mathbb{Z}^+$ and perceptibly the equation in which solutions to be evaluated is converted into ${X_1}^2 - 5X_1Y_1 + 4{Y_1}^2 + C_1X_1 = 0$ where $C_1 = C/2$. By the argument as explained above, $X_1|{Y_1}^2$ and hence ${Y_1}^2 = X_1Z_1$ for some $Z_1 \in \mathbb{Z}^+$.

Again, if $p|X_1$ and $p|Z_1$, then $p|Y_1$ and the precise relation $X_1 - 5Y_1 + 4Z_1 + C_1 = 0$ is also true for all $X_1, Y_1 \in \mathbb{Z}^+$.

Carrying on this procedure till the equation ${X_n}^2 - 5X_nY_n + 4{Y_n}^2 + X_n = 0$ is reached.

Further if $p|X_n$ and $p|Z_n$, then $p|Y_n$ and the accurate equation $X_n - 5Y_n + 4Z_n + 1 = 0$ is detected for all $X_n, Y_n \in \mathbb{Z}^+$.

Finally, p divides 1 which is impossible.

As a result, our supposition that X and Z have common divisors is erroneous. This shows that that $gcd(X, Z) = 1$.

Thus, by the necessary and sufficient condition that the product two coprime numbers should be a perfect square if and only if each of them is a perfect square, $X = P^2$ and $Z = Q^2$ where $P, Q \in \mathbb{Z}^+$ and $\gcd(P, Q) = 1$.

These adoptions of X and Z provides that $Y = PQ$ and subsequently the essential equation can be developed into $P^2 - 5PQ + 4R^2 + C = 0$.

By Corollary 2.1, the values of P, Q and C are searched by $(P, Q, C) = (J_{2n+1}, J_{2n-1}, 2^{2n-1})$ and therefore $(X, Y, C) = \left({J_{2n+1}}^2, J_{2n-1}J_{2n+1}, 2^{2n-1}\right), n \geq 1$.

Corollary: 5.1

The non-negative integer solutions for the equation $X^2 - 5XY + 4Y^2 - CX = 0$ are symbolized by $(X, Y, C) = \left({J_{2n+2}}^2, J_{2n+2}J_{2n}, 2^{2n}\right)$ where $n \geq 0$.

Theorem: 6

(i) The patterns of positive integer solutions to the equation $X^2 - 5XY + 4Y^2 + CY = 0$ are incarnated by $(X, Y, C) = \left(J_{2n-1}J_{2n+1}, {J_{2n-1}}^2, 2^{2n-1}\right)$ where $n \geq 1$.

(ii) The infinitely several positive integer solutions to the equation $X^2 - 5XY + 4Y^2 - CY = 0$ are signified by $(X, Y, C) = \left(J_{2n+2}J_{2n}, {J_{2n}}^2, 2^{2n}\right)$ where $n \geq 0$.

Theorem: 7

The feasible solution in Jacobsthal-Lucas numbers for two unlike binary quadratic equations $X^2 - XY - 2Y^2 = 9C$ and $X^2 - XY - 2Y^2 = -9C$ are presented by $(X, Y, C) = (j_{2n}, j_{2n-1}, 2^{2n-1})$, $n \geq 1$ and $(X, Y, C) = (j_{2n+1}, j_{2n}, 2^{2n}), n \geq 0$ respectively.

Theorem: 8

Let X, Y be two distinct natural numbers.

(i) If $X^2 - 5XY + 4Y^2 = 9C$, then $(X, Y, C) = (j_{2n+1}, j_{2n-1}, 2^{2n-1})$, $n \geq 1$.

(ii) If $X^2 - 5XY + 4Y^2 = -9C$, then $(X, Y, C) = (j_{2n+2}, j_{2n}, 2^{2n})$, $n \geq 0$.

Theorem: 9

If X , Y be any two non-negative integers such that $X^2 - XY - 2Y^2 + 9CX = 0$, then either $(X, Y, C) = \left(9{J_{2n}}^2, 9J_{2n}J_{2n-1}, 2^{2n}\right), n \geq 1$ or $(X, Y, C) = \left({j_{2n+1}}^2, j_{2n+1}j_{2n}, 2^{2n}\right)$, $n \geq 0$.

Proof:

Assume that $X^2 - XY - 2Y^2 + 9CX = 0$ for some non-negative integers X and Y.

If $9|X$, then $9|Y \Rightarrow X = 9U$ and $Y = 9V$ for some $U, V \in \mathbb{Z}^+$

Therefore, the needed equation in two unknowns X and Y is enhanced in terms of U and V as $U^2 - UV - 2V^2 + CU = 0$.

By theorem 4, $(U, V, C) = \left({J_{2n}}^2, J_{2n}J_{2n-1}, 2^{2n}\right) \Rightarrow (X, Y, C) = \left(9{J_{2n}}^2, 9J_{2n}J_{2n-1}, 2^{2n}\right)$.

If $9 \nmid X$, then by theorem [II], $X = U^2$ and $Y = UV$.

These choices of U and V simplifies the considered equation into $U^2 - UV - 2V^2 + 9C = 0$.

By theorem 7, it is resolved that

$(U, V, C) = (j_{2n+1}, j_{2n}, 2^{2n}) \Longrightarrow (X, Y, C) = \left({j_{2n+1}}^2, j_{2n+1}j_{2n}, 2^{2n}\right)$ where $n \geq 0$.

Conversely if $(X, Y, C) = \left(9{J_{2n}}^2, 9J_{2n}J_{2n-1}, 2^{2n}\right)$, then by the implementation of corollary 1.2

$$\begin{aligned} X^2 - XY - 2Y^2 + 9CX &= \left(9{J_{2n}}^2\right)^2 - \left(9{J_{2n}}^2\right)(9J_{2n}J_{2n-1}) - 2(9J_{2n}J_{2n-1})^2 + 9C\left(9{J_{2n}}^2\right) \\ &= 81{J_{2n}}^2\left\{{J_{2n}}^2 - J_{2n}J_{2n-1} - {J_{2n-1}}^2 + C\right\} = 0, \end{aligned}$$

Likewise, the very same equation might well be fulfilled for $(X, Y, C) = \left({j_{2n+1}}^2, j_{2n+1}j_{2n}, 2^{2n}\right)$.

Theorem: 10

Let $X, Y \in \mathbb{Z}^+$, the set of all positive integers.

(i) If $X^2 - XY - 2Y^2 - 9CX = 0$, then either $(X, Y, C) = \left(9{J_{2n+1}}^2, 9J_{2n+1}J_{2n}, 2^{2n}\right), n \geq 0$ or $(X, Y, C) = \left({j_{2n}}^2, j_{2n}j_{2n-1}, 2^{2n-1}\right)$, $n \geq 1$.

(ii) If $X^2 - XY - 2Y^2 + 9CY = 0$, then either $(X, Y, C) = \left(9J_{2n}J_{2n-1}, 9{J_{2n-1}}^2, 2^{2n-1}\right), n \geq 1$ or $(X, Y, C) = \left(j_{2n}j_{2n+1}, {j_{2n+1}}^2, 2^{2n}\right)$, $n \geq 0$.

(iii) If $X^2 - XY - 2Y^2 - 9CY = 0$, then either $(X, Y, C) = \left(9J_{2n+1}J_{2n}, 9{J_{2n}}^2, 2^{2n}\right)$, $n \geq 0$ or $(X, Y, C) = \left(j_{2n}j_{2n-1}, {j_{2n-1}}^2, 2^{2n-1}\right)$, $n \geq 1$.

Theorem: 11

Let X, Y be two distinct non-negative integers. Then

(i) The two different set non-negative integer solutions to the equation $X^2 - 5XY + 4Y^2 + 9CX = 0$ are discovered by $(X, Y, C) = \left(9{J_{2n+1}}^2, 9J_{2n+1}J_{2n-1}, 2^{2n-1}\right)$, $n \geq 1$ and $(X, Y, C) = \left({j_{2n+2}}^2, j_{2n+2}j_{2n}, 2^{2n}\right), n \geq 0$.

(ii) All possible solutions in Jacobsthal and Jacobsthal -Lucas numbers to the equation $X^2 - 5XY + 4Y^2 - 9CX = 0$ are determined by $(X, Y, C) =$

$\left(9{J_{2n+2}}^2, 9J_{2n}J_{2n+2}, 2^{2n}\right), n \geq 0$ and $(X, Y, C) = \left({j_{2n+1}}^2, j_{2n+1}j_{2n-1}, 2^{2n-1}\right), n \geq 1$.

(iii) If $X^2 - 5XY + 4Y^2 + 9CY = 0$, then two sequences of solutions are presented by $(X, Y, C) = \left(9J_{2n-1}J_{2n+1}, 9{J_{2n-1}}^2, 2^{2n-1}\right), n \geq 0$ and $(X, Y, C) = \left(j_{2n}j_{2n+2}, {j_{2n}}^2, 2^{2n-1}\right), n \geq 1$.

(iv) If $X^2 - 5XY + 4Y^2 - 9CY = 0$, then either $(X, Y, C) = \left(9J_{2n}J_{2n+2}, 9{J_{2n}}^2, 2^{2n}\right)$, $n \geq 1$ or $(X, Y, C) = \left(j_{2n-1}j_{2n+1}, {j_{2n-1}}^2, 2^{2n-1}\right), n \geq 0$.

CONCLUSION:

In this article, the generic solutions to a specific set of unambiguous binary quadratic equations are revealed in terms of Jacobsthal and Jacobsthal-Lucas numbers. In this manner, one may explore solutions to some cubic or higher degree Diophantine equations having more than two variables concerning other periodic sequences of integers.

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