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FINITE AND NEIGHBORHOOD GAUSSIAN PRIME DISTANCE GRAPHS

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Abstract
The prime distance is a measure of the connectivity of a graph based on the concept of prime numbers. The Gaussian prime distance (GPD) graph G is defined as any graph with two relatively prime vertices. The Gaussian neighborhood prime distance (GNPD) graph is made up of a graph G and each vertex v, with neighboring vertices being relatively prime numbers. It is formed by dividing G into n separate Gaussian numbers such that adjacent vertex numbers are relatively prime numbers. The order in the XY plane was structured linearly by means of Gaussian integers to create a spiral structure and demonstrate that certain graph family have a finite Gaussian prime distance with the Gaussian integers.
Keywords: Finite Gaussian Prime distance, Neighborhood Gaussian Prime Distance, Spiral order, Gaussian integers.
AMS Subject Classification (2010): 05C78,05C76

1. INTRODUCTION

Labeling in graph theory has been applied in many research fields. For instance, in social network analysis, labeling has been used to identify influential nodes or communities. Additionally, in bioinformatics, graph labeling has been applied to analyze protein structures and DNA sequences. Since the introduction of complex numbers and their intrinsic properties, many wonders have happened in various fields. The idea of prime labeling for graphs was urban by Roger Entringer and original published in a work by Tout et al. [5] in the early 1980s. Since then, it has become a popular topic of study among academics. The readers may consult Gallian[1] for a complete list of results about prime graphs. In this document, we introduce the idea of neighborhood-prime labeling by Gaussian integers, which is inspired by the reading of prime labeling in Gaussian integers. We anticipate that many academics will find this to be an exciting topic of research in the future. Finite Gaussian prime distance graphs and neighborhood prime distance graphs have vertices with substantially prime distances, and neighborhood vertices are relatively prime if every vertex of a graph G be able to label by n distinct numbers. Walks, trails, paths, double star trees, caterpillars, spider webs, snake graphs, fans, and many other graph families are known to allow prime distance labeling [1, 2]. In this study, the concept of finite Gaussian prime distance was generalized to complex numbers with both real and imaginary components that are integers, sometimes known as Gaussian integers..

This paper first discusses the first n Gaussian integers in this study before investigating the finite Gaussian prime distance graph and neighborhood prime distance graph for the Gaussian integers. The study then presents the results of the analysis of these graphs, including their properties and characteristics, which provide insights into the distribution and behavior of prime numbers in the Gaussian integer domain. The notion of Gaussian integers is discussed in Section 2. Many familiar property of the ordering on natural N, as well as additional properties discussed, are preserved in this XY plane spiral order. Section 3 demonstrates how spiral ordering can lead to some tree families being able to accept prime distance labels with Gaussian integers, as well as snake graph families accepting neighborhood prime labeling. We also discuss an alternative graph in which prime distance and neighborhood prime distance can be accommodated.

2 A. BACKGROUND AND DEFINITIONS

To build the groundwork for our approach, we begin with some understanding of Gaussian integers. The Gaussian integer is a number in the form $\alpha + \beta i$, assuming that α , β are integers and that i is the imaginary unit of the complex number. [5,7]

The Gaussian integers, $Z[i] = \{\alpha + \beta i / \alpha, \beta \in Z\}$ where $i^2 = -1$

One of $(\pm 1, \pm i)$ is called the unit and $N(\alpha + \beta i)$ is the norm of a GI $\alpha + \beta i$, is given by $\alpha^2 + \beta^2$. [5,7]

Definition 2.1

A Gaussian integer $\tau \in Z[i]$ is prime if the divisors include only the units or $\pm\tau, \pm\tau i$ [5,7]. Aside from this definition, Gaussian primes are characterized by the following. [3]

Theorem 2.2 (Bezout's theorem)

Let δ be any gcd of two non-zero Gaussian integers α and β, followed by $\delta = \alpha x + \beta y$ for some $x, y \in Z[i]$

Corollary 2.3

$1 = \alpha x + \beta y$ *for some* $x, y \in Z[i]$ *if and only if Gaussian integers* α *and* β *as relatively primes*

Theorem 2.4[7]

A Gaussian integer $\alpha + \beta i \in Z[i]$ is a Gaussian prime [5,7] if and only if either

- $\alpha + \beta i = \pm(1 + i)$
- $N(\alpha + \beta i)$ is a prime integer congruent to 1 (mod 4), or
- One of α, β, is zero, and $|\alpha| \equiv 3(mod\ 4) or |\beta| \equiv 3(mod\ 4)$
- If $\alpha \neq 0\ and\ \beta \neq 0$ and $\alpha^2 + \beta^2$ is a prime integer that is not congruent to $3(mod\ 4)$

Definition 2.5[7]

Let the two Gaussian integers$\rho_1\ and\ \rho_2$are comparatively prime or co-prime if they only have the units in Z[i] as their common divisors.

2.B. GRAPH LABELING WITH GAUSSIAN INTEGERS AS PRIMES

There will be no loops or numerous edges in our graphs. The goal of our study is extended to Gaussian integers in the finite Gaussian prime distance labeling [1]. To determine what counts as the first n Gaussian integers[5], we must first define "the first n Gaussian integers". This proves that Gaussian numbers have the same properties as real numbers

Here is our proposed ordering.

Definition 2.6

Using recursive definitions of the Gaussian integers, we can recursively order them in spiral order. As with, the spiral ordering denotes the nth Gaussian integer by τ_n.Equation 1 states that the spiral is ordered from $\tau_1 = 1$ to base on the nth integer in the spiral. See Fig.1 for an illustration.

The first 10 Gaussian integers are ordered according to this ordering [3]

$$1, 1+i, i, -1+i, -1, -1-i, -i, 1-i, 2-i, 2 \dots \dots \dots$$

In the spiral ordering, we write $[\tau_k]$ to represent all the k[th] Gaussian integers.

$$\tau_{k+1} = \begin{cases} \tau_k + i & if \ |\mathrm{Re}(\tau_k)| > |\mathrm{Im}(\tau_k)|, \ \ \mathrm{Re}(\tau_k) > 0 \ \ and \ \ -(\mathrm{Re}(\tau_k) - 1) \le \mathrm{Im}(\tau_k) \le (\mathrm{Re}(\tau_k) - 1) \\ \tau_k - 1 & if \ |\mathrm{Re}(\tau_k)| \le |\mathrm{Im}(\tau_k)|, \ \ \mathrm{Im}(\tau_k) > 0 \ \ and \ \ -\mathrm{Im}(\tau_k) \le \mathrm{Re}(\tau_k) \le \mathrm{Im}(\tau_k) \\ \tau_k - i & if \ |\mathrm{Re}(\tau_k)| \ge |\mathrm{Im}(\tau_k)|, \ \ \mathrm{Re}(\tau_k) < 0 \ \ and \ \ \mathrm{Re}(\tau_k) < \mathrm{Im}(\tau_k) < -\mathrm{Re}(\tau_k) \\ \tau_k + 1 & if \ |\mathrm{Re}(\tau_k)| \le |\mathrm{Im}(\tau_k)|, \ \ \mathrm{Im}(\tau_k) < 0 \ \ and \ \ \mathrm{Im}(\tau_k) \le \mathrm{Re}(\tau_k) \le -\mathrm{Im}(\tau_k) \end{cases}$$

Gaussian integer spiral ordering leads to finite Gaussian prime distance labeling of other Gaussian integer graphs.

Definition2.7

Let $G = (V, E)$. A finite GPD labeling graph of Gaussian integers G is a one-to-one labeling $\varphi : V(G) \to Z[i]$ such that if $uv \in E(G)$, then $\varphi(u)$ and $\varphi(v)$are relatively prime to each other. So G is called a finite Gaussian prime distance graph if there exists a finite Gaussian prime distance labeling exists.

Prime distance labeling requires distinct labels at each vertex of G.

2.C. AN OVERVIEW OF SPIRAL ORDERING PROPERTIES

An order of Gaussian spiral is defined here in several pieces. Spiral corners occur when spirals move from one direction to another direction.. Spirals have branches when they travel in a straight line north, south, east, or west.

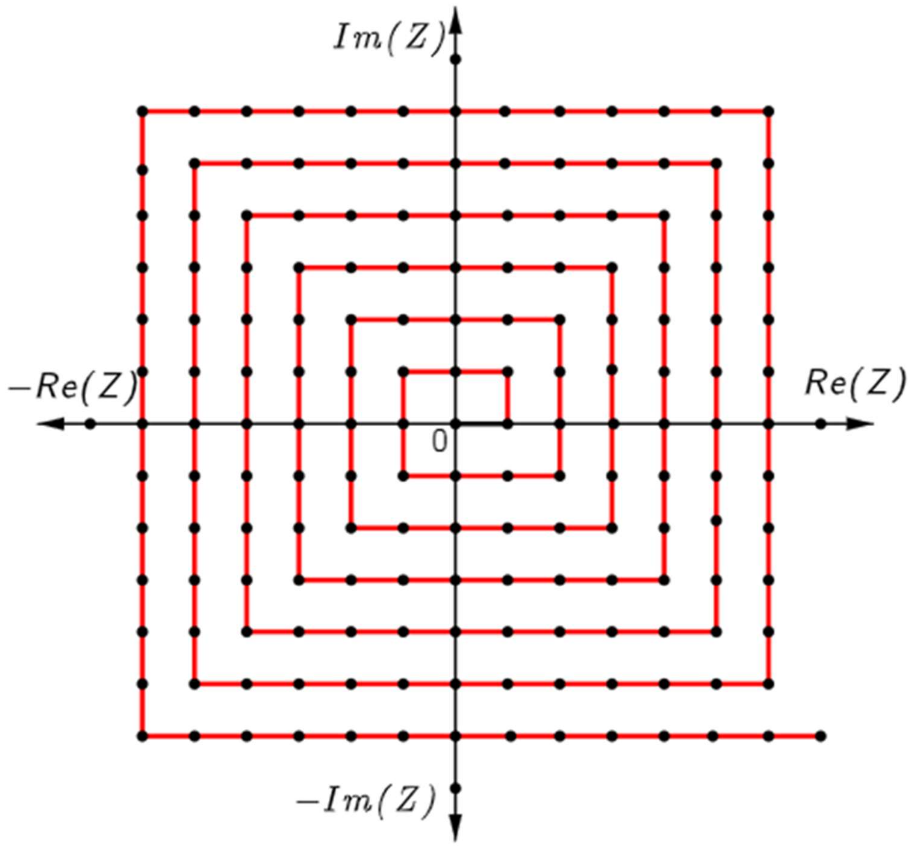

Fig.2. The spiral order of Gaussian Integers

Our first objective is to decide the ordinal numeral of an inconsistent Gaussian whole number, $a + bi$ in the winding request in light of which kind of branch or corners it lies on.
Let $O_{GI}(a + bi)$be used to mean the ordinal numeral of $a + bi$ in the twisting requesting.
The points lie on the axes are

On X axis
Either $\mathrm{Re}(z) > 0$ *and* $\mathrm{Im}(z) = 0$ *or* $\mathrm{Re}(z) < 0$ *and* $\mathrm{Im}(z) = 0$
On Y axis
Either $\mathrm{Re}(z) = 0$ *and* $\mathrm{Im}(z) > 0$ *or* $\mathrm{Re}(z) = 0$ *and* $\mathrm{Im}(z) < 0$

I and III quadrant corners are the Gaussian integers such that their real and imaginary parts are equal and of same in signs. II quadrant corners are having equal real and imaginary part and are of opposite signs.
IV quadrant corners are of opposite in signs and $\mathrm{Re}(z) = |\mathrm{Im}(z)| + 1$

Lemma 2.8.
Ordinal numerals of four types of points lie on X – axes, and Y-axes and are found as follows:

$$O_{GI}(a+bi)=\begin{cases}4a^2-3a & \text{if } a>0 \text{ and } b=0\\ 4b^2-b & \text{if } a=0 \text{ and } b>0\\ 4a^2-a & \text{if } a<0 \text{ and } b=0\\ 4b^2-3b & \text{if } a=0 \text{ and } b<0\end{cases}$$

Proof

The ordinal numerals of a GI are the number of points in its path. First, we'll examine the points on the positive side of the X-axis. We started with the very first integer $1 + 0i$, and its ordinal numerals are 1. The second point in the axis is $2 + 0i$, it finally crosses a 3 x 3 grid of points having three columns of three points each, and it may cross another point to reach its ordinal numerals. For our convenience in the calculation, we include point zero and exclude point $2 + 0i$. The third point in the axis is $3 + 0i$,which crosses a grid of five columns of five points each, and it may cross another two points to reach its position.

By progressing in this fashion, the point $a + bi, b = 0$ on the positive real axis is crossed by "$2a - 1$" columns of "$2a - 1$" a point each and it may crosses another "$a - 1$" points. It crossed completely $(2a - 1)(2a - 1) + (a - 1) = 4a^2 - 4a + 1 + a - 1 = 4a^2 - 3a$ points. So, its ordinal numerals are$4a^2 - 3a$

Second, we'll look at the points on the positive side of the Y axis. We started with the first point $0+i$ in the Y axis, which ordinal number is 3. The second point in the axis is;$0+2i$, from which crosses three columns with four points each and it may cross another two points to reach its place. We use the current y-axis as the point instead of zero as the center point. $0 + 3i$, the Y axis third point, crosses five columns of six points each and it may cross another three points.

In general, the positive side of Y axis point $0 + bi, b > 0$ is crossed by "2b-1" columns of 2a points each by continuing in this manner. It entirely crossed $(2b - 1)(2b) + b = 4b^2 - 2b + b = 4b^2 - b$ points.So, their ordinal numerals are$4b^2 - b$

Similarly, we get the indices of the points on the negative sides of the X and Y axes.

Lemma 2.9

Ordinal numerals of the corners lie on the four quadrants are found as follows:

$$O_{GI}(a+bi)=\begin{cases}4a^2-2a & \text{if corners lie in I quadrant}\\ 4a^2 & \text{if corners lie in I quadrant}\\ 4a^2+2a & \text{if corners lie in III quadrant}\\ (a-b)^2 & \text{if corners lie in IV quadrant}\end{cases}$$

Proof

First, we'll examine the XY plane's first quadrant corner points. In the first quadrant, we started with the very first corner integer $1 + i$. The second corner point in the first quadrant, $2 + 2i$, crosses three rows of four points each, whilst the third corner point in the first quadrant, $3 + 3i$, crosses seven rows of eight points each.

By progressing in this fashion, the first quadrant's corner point $a + bi, a = b$ is crossed by "$2a - 1$" rows of $2a$ points each. It crossed completely $2a\ (2a - 1) = 4a^2 - 2a$ points.

Second, we'll look at the second quadrant corner points in the XY plane. We started with the first corner integer $-1+i$ in the second quadrant, which crosses two columns with two points each. We use the current corner point instead of zero as the center point. $-2+2i$, the second quadrant's second corner point, crosses four columns of four points each, whereas $-3+3i$, the third quadrant's third corner point, crosses six columns of six points each.
The second quadrant's corner point $-a+ai, a=b$ is crossed by "2a" columns of 2a points each by continuing in this manner. It entirely crossed $2a\ (2a) = \ 4a^2$ points.
Third, we'll look at the third quadrant corner points of the XY plane. We began in the third quadrant with the first corner integer, $-1-i$, which crosses two rows of three points each. We use the current corner point instead of zero as the center point. -2 $-\ 2i$, the third quadrant's second corner point, crosses four rows of five points each, whereas $-3-3i$, the third quadrant's third corner point, crosses six rows of seven points each.
The third quadrant's corner point $-a-ai, a=b$ is crossed by "2a" rows of $2a+1$ points each by continuing in this manner. It completely crossed $2a\ (2a+1) = \ 4a^2+2a$ points.
Finally, we'll look at the XY plane's fourth quadrant corner points. In the fourth quadrant, we started with the first corner integer 2-i, which crosses three columns with three points each. Instead of zero, we use the current corner point as the center point. The second corner point of the fourth quadrant, 3-2i, crosses five columns of five points each, whereas the third corner point, 4-3i, crosses seven columns of seven points each.
By proceeding in this fashion, the fourth quadrant's corner point $a+bi, a>b$ is crossed by "$(a-b)$" columns of $(a-b)$ points each. It completely crossed the $(a-b)(a-b) = (a-b)^2$points.

3.A. FINITE GAUSSIAN PRIME DISTANCE RESULTS FOR GRAPH FAMILIES

Definition 3.1

A lattice graph is a type of graph that represents a structure consisting of points or nodes arranged in a regular, grid-like pattern. It is commonly used in physics and mathematics to model crystal structures and other regular arrangements of particles.

Theorem 3.2

Every grid graph is a finite Gaussian prime distance graph.

Proof

The set of Gaussian integers we are considering here is already on a grid system in the XY plane, and we know of consecutive Gaussian integers in the system that differ by a unit difference and unit distance points having the gcd of 1. By definition 2.4. every Lattice graph is a finite Gaussian prime distance graph
Example: A 2x2 grid graph.

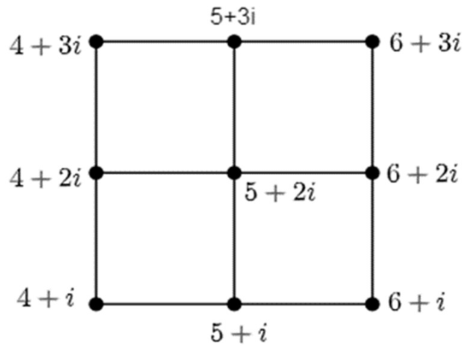

Fig.2. The 2x2 grid graph

$$5+3i=(1)\ (4+3i)+(1)$$
$$4+3i=(1)\ (4+3i)+(0)$$
$$Thus, \gcd\ (4+3i, 5+3i)=1$$

Similarly,

$$4+3i=(1)\ (4+2i)+(i)$$
$$4+2i=(i)\ (2-4i)+(0)$$
$$Thus, \gcd\ (4+3i, 4+2i)=i$$

$$5+3i=(1)\ (4+2i)+(1+i)$$
$$4+2i=(1+i)\ (3-i)+(0)$$
$$Thus, \gcd\ (5+3i,\ 4+2i)=1+i$$

Definition 3.3

The path graph,[1,5]P_n, on n vertices is the graph with $V(P_n) = \{V_1, V_2, V_3, \ldots\ldots V_n\}$ and $E(P_n) = \{\ V_j V_{j+1}: 1 \leq j \leq n-1\}$(Fig.2)

Theorem 3.4

Every path graph is a finite Gaussian prime distance graph.

Proof

A square grid graph is a particular kind of graph that has edges joining adjacent vertices and vertices arranged in a square grid pattern.We can say that the square grid graph is a composition of multiple path graphs if we assume of the vertical and horizontal lines of the grid as path graphs, which implies that they enhance the property of finite prime distance. It therefore proves.

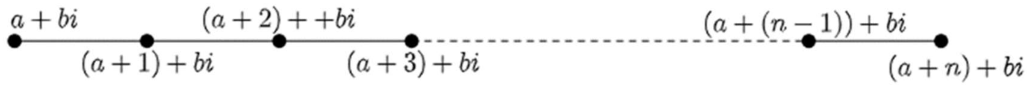

Fig.3.The path graph

Definition 3.5

Grid graphs are bipartite, meaning that all edges connect a vertex in one set to a vertex in the other set.

Theorem 3.6

Bipartite graphs admits finite Gaussian prime distance labeling.

Proof

If we color the vertices of the grid graphs in a checkerboard fashion, it can be verified easily.

Refer Fig 4

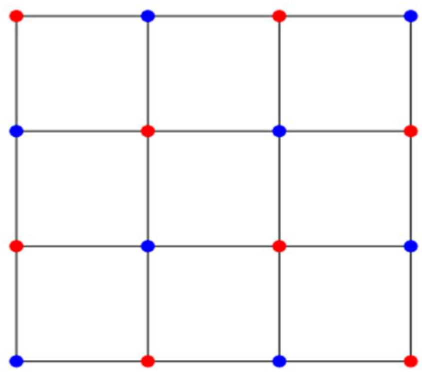

Fig.4. A 4 x 4 Grid graph as a bipartite graph

Definition 3.7. The star with central vertex and leaves through , with an edge connecting each subsequent pair of vertices and where , is the Dutch windmill graph or friendship graph. As a result, has n copies of joined at [1,6]. is seen in Figure 3.

Theorem 3.8

Any Dutch windmill graph admits a finite Gaussian prime distance labeling.

Proof

The Dutch windmill graph is a simple graph consisting of a central vertex v_0, and n blades, each containing m vertices, where every vertex in the ith blade is adjacent to v_0 and to the corresponding vertex in the (i+1)th blade. Let us start with vertex $v_0 = \alpha + \beta i, v_0 \in Z[i]$, the center vertex of the Dutch windmill graph. Assign a vertex value in the blade neighboring to v_0 such that v_0 *and* v_i are relatively prime Gaussian integers.

Hence, every adjacent vertex in the Dutch windmill graph is relatively prime and admits a finite prime distance.

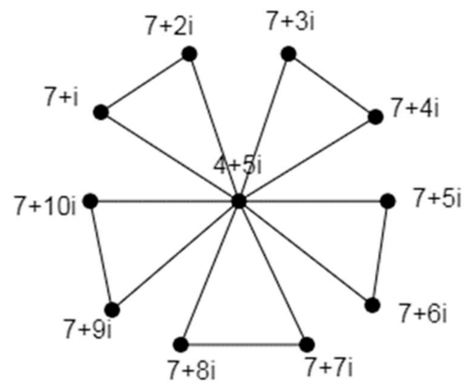

Fig.5. The Dutch windmill graph

Theorem 3.9

Any star graph   is a finite Gaussian prime distance graph.

Proof

If we start with a star graph, we have one central vertex connected to several peripheral vertices. Labeling the central vertex v_0 with $\alpha + \beta i \in Z[i]$ and labeling all other peripheral vertices with the Gaussian integers relatively prime to v_0. Hence, every adjacent vertex in the star graph is relatively prime and admits a finite prime distance.

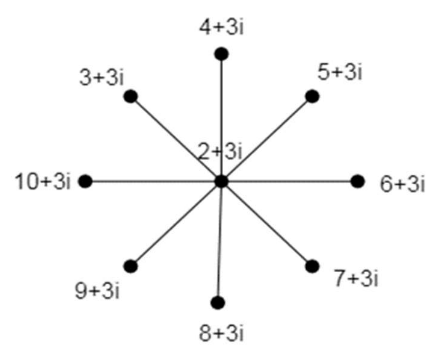

Fig.6. The star graph

Definition 3.10

The (n, r, s)-double star tree**,** where n, r, s be integers with $r \leq s$ then $DS_{n,r,s}$, is the graph with
$V(DS_{n,r,s}) = \{v_1, \dots, v_n,\ v_{n+1}, \dots, v_{n+r-1}, v_{n+r}, \dots, v_{n+r+s-2}\}$, and
$E(DS_{n,r,s}) = \{v_j v_{j+1}: 1 \leq j \leq n-1, v_1 v_{n+j}: 1 \leq j \leq r-1, v_n v_{n+r+j}: 0 \leq j \leq s-2\}$

Theorem 3.11

Any (n, r, s) *−double star tree admits a finite Gaussian prime distance labeling.[1,5]*

Proof

A path graph is a graph in which all vertices are associated in a linear sequence, while a star graph has one central vertex connected to all other vertices. A double star tree has a path of distance end to end n, with end vertices and v_1 and v_n being the middle vertices for stars having r and s vertices. Theorems 3.4 and 3.9 state that every path graph and every star graph admit a finite Gaussian prime distance, making them finite Gaussian distance graphs.

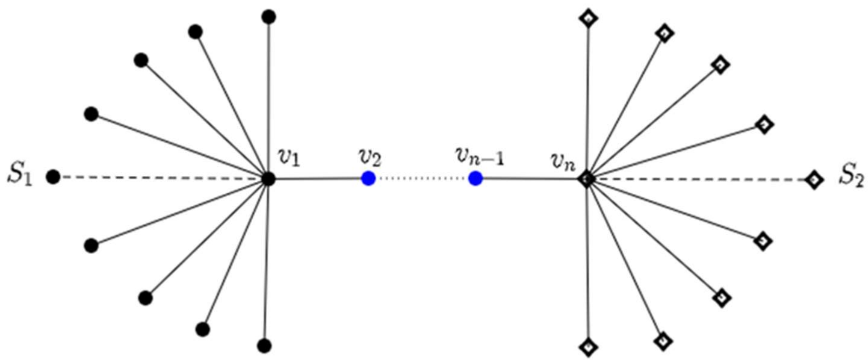

Fig.7. The (n,r,s)-double star tree

Definition 3.12

A tree with at least one vertex with a degree of three and all other vertices with degrees of one or two is called a spider graph. (Fig.8)

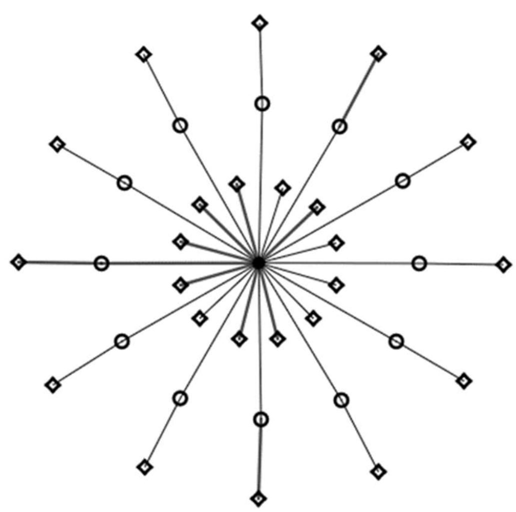

Fig.8 The Gaussian prime distance spider graph

Theorem 3.13

Any spider tree admits finite Gaussian prime distance labeling.

Proof

Let T be a spider tree and assume the center vertex v_0 has degree n. Label the any vertex $v_k, 1 \le k \le n$ with the Gaussian integers relatively prime to v_0 and continue the process to the adjacent vertices of $v_k, 1 \le k \le n$.

3.B. NEIGHBORHOOD PRIME DISTANCE RESULTS FOR GRAPH FAMILIES

Definition 3.14

The neighborhood of a vertex v in Graph G is the set of all vertices in G that are neighboring to v, and it is represented by the number N(v).

Definition 3.15

Let G = (V (G), E(G)) be an n-vertex graph. If for every vertex $v \in V(G)$ with deg(v) > $1, gcd\,\{f(u): u \in N(v)\} = 1$, then $f: V(G) \to Z[i]$ is said to be neighborhood-prime labeling.

Definition 3.16

A triangular cactus is a linked graph that only consists of triangles as its building blocks.

Remark

A prime graph may or may not also be a neighborhood-prime graph and vice versa. This independence between the words "prime graphs" and "neighborhood-prime graphs" refers to their flexibility.

Theorem 3.17

Every path graph Pn admits neighborhood prime labeling.

Proof

Let v_i be any Gaussian integer in P_n where $v_i = \alpha + \beta i \in Z[i]$. Already we say that every grid graph (Theorem 3.2) and every path P_n(Theorem 3.4) admits finite prime labeling. If we take the neighborhood of v_i as $v_{i-1} and\ v_{i+1}$ belongs to the set $\{(\alpha-1)+\beta i, (\alpha+1)+\beta i, \alpha+(\beta-1)i, \alpha+(\beta+1)i\}$, Clearly,$v_i, v_{i-1} and\ v_{i+1}$ are relatively prime to each other.

Theorem 3.18

Every triangular cactus graph is a neighborhood prime labeling graph.

Proof

Identifying a path as the cactus graph's basis is the first step in proving the theorem. According to Theorem 3.18, the path that is derived from the grid graph allows for neighborhood prime labeling. Since their differences in the path alternative are 2i, it follows that their gcd is -i. Gaussian integers are relatively prime in this case. Thus every triangular cactus graph admits neighborhood prime labeling. [Fig.9]

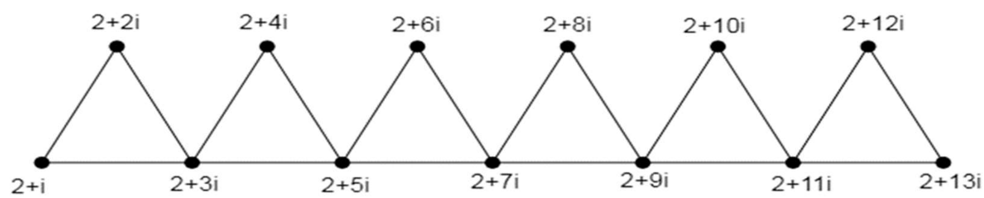

Fig 9. Triangular Cactus Graph with 13 vertices

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